NDA/NA

National Defence Academy/Naval Academy

SOLVED PAPER 2019 (II)

PAPER I : Mathematics

1. If both p and q belong to the set $\{1, 2, 3, 4\}$, then how many equations of the form $px^2 + qx + 1 = 0$ will have real roots?

(a) 12 (b) 10
(c) 7 (d) 6

(*d*) Equation $px^2 + qx + 1 = 0$, has real roots, where p and q belong to the set $\{1, 2, 3, 4\}$.

$\therefore \quad q^2 - 4p \geq 0$

[$\because$ for real roots of a quadratic equation $b^2 - 4ac \geq 0$]

It is possible if value of $(p, q) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4)$ and $(3, 4)$

Hence, the number of equations are 6.

2. What is the value of $1 - 2 + 3 - 4 + 5 - \dots + 101$?

(a) 51 (b) 55
(c) 110 (d) 111

(*a*) Given series,

$= 1 - 2 + 3 - 4 + 5 - \dots + 101$
$= (1 + 3 + 5 + \dots + 101) - (2 + 4 + 6 + \dots + 100)$
$= (1 + 3 + 5 + \dots$ 51 terms$) - (2 + 4 + 6 + \dots$ 50 terms$)$
$= \frac{51}{2}[2 + (51 - 1) \times 2] - \frac{50}{2}[4 + (50 - 1) \times 2]$

[$\because$ both series are AP and $S_n = \frac{n}{2}[2a + (n-1)d]$]

$= \frac{51}{2} \times 102 - \frac{50}{2} \times 102$
$= 2601 - 2550 = 51$

3. If A, B and C are subsets of a given set, then which one of the following relations is not correct?

(a) $A \cup (A \cap B) = A \cup B$
(b) $A \cap (A \cup B) = A$
(c) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
(d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

(*a*) Let U be the set and A, B and C are the subset of U.
We know that, $A \cup (A \cap B) = A$,
So option (a) is not correct.
$A \cap (A \cup B) = A$, so option (b) is correct.
$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$, so option (c) is correct.
and $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ so option (d) is correct.

4. If the sum of first n terms of a series is $(n + 12)$, then what is its third term?

(a) 1 (b) 2
(c) 3 (d) 4

(*a*) Sum of first n term of a series $= n + 12$
$\Rightarrow a_1 + a_2 + a_3 + \dots + a_n = n + 12$
Put $n = 1$, $a_1 = 1 + 12 = 13$
Put $n = 2$, $a_1 + a_2 = 2 + 12 \Rightarrow a_1 + a_2 = 14$
$\Rightarrow 13 + a_2 = 14 \Rightarrow a_2 = 1$
Put $n = 3$
$a_1 + a_2 + a_3 = 3 + 12$
$\Rightarrow \quad 13 + 1 + a_3 = 15$
$\Rightarrow \quad a_3 = 15 - 14 = 1$

5. What is the value of k for which the sum of the squares of the roots of $2x^2 - 2(k-2)x - (k+1) = 0$ is minimum?

(a) -1 (b) 1
(c) $\frac{3}{2}$ (d) 2

(*c*) Let α, β be the roots of equation.
$2x^2 - 2(k-2)x - (k+1) = 0$

$\because \alpha + \beta = \frac{2(k-2)}{2} = k - 2,$

$\alpha\beta = \frac{-(k+1)}{2}$

We know that

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= (k-2)^2 + 2 \times \frac{k+1}{2}$
$= k^2 + 4 - 4k + k + 1$
$= k^2 - 3k + 5$
$= k^2 - 3k + \frac{9}{4} - \frac{9}{4} + 5$
$= \left(k - \frac{3}{2}\right)^2 + \frac{11}{4}$

$\alpha^2 + \beta^2$ is minimum, if $\left(k - \frac{3}{2}\right) = 0$

$\Rightarrow \quad k = \frac{3}{2}$

6. If the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, then which one of the following is correct?

(a) a, b and c are in AP
(b) a, b and c are in GP
(c) a, b and c are in HP
(d) a, b and c do not follow any regular pattern

(*c*) The roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal.

$\therefore b^2(c-a)^2 - 4a(b-c).c(a-b) = 0$

[$\because ax^2 + bx + c = 0$ of roots are real if $b^2 - 4ac \geq 0$

$\Rightarrow b^2(c^2 + a^2 - 2ca) - 4ac\,(ab - b^2 - ac + bc) = 0$

$\Rightarrow b^2c^2 + a^2b^2 - 2ab^2c - 4a^2bc$
$+ 4ab^2c + 4a^2c^2 - 4abc^2 = 0$
$\Rightarrow b^2c^2 + a^2b^2 + 2ab^2c$
$- 4a^2bc - 4abc^2 + 4a^2c^2 = 0$
$\Rightarrow b^2(c^2 + a^2 + 2ac) - 4abc\,(a + c)$
$+ 4a^2c^2 = 0$
$\Rightarrow b^2(c + a)^2 - 4abc\,(a + c) + (2ac)^2 = 0$
$\Rightarrow [b(c + a) - 2ac]^2 = 0$
$\Rightarrow b\,(c + a) - 2ac = 0$
$\Rightarrow b(c + a) = 2ac \Rightarrow b = \dfrac{2ac}{c + a}$

So, a, b and c are is HP.

7. If $|x^2 - 3x + 2| > x^2 - 3x + 2$, then which one of the following is correct?

(a) $x \le 1$ or $x \ge 2$ (b) $1 \le x \le 2$
(c) $1 < x < 2$
(d) x is any real value except 3 and 4

(**c**) $|x^2 - 3x + 2| > x^2 - 3x + 2$
$\Rightarrow -(x^2 - 3x + 2) > x^2 - 3x + 2$
[if $x^2 - 3x + 2 < 0$, and $x^2 - 3x + 2 > 0$ not possible]
$\Rightarrow -2(x^2 - 3x + 2) > 0$
$\Rightarrow x^2 - 3x + 2 > 0$
$\Rightarrow x^2 - 2x - x + 2 > 0$
$\Rightarrow (x - 2)(x - 1) > 0$
$\therefore 1 < x < 2$ is correct.

8. A geometric progression (GP) consists of 200 terms. If the sum of odd terms of the GP is m, and the sum of even terms of the GP is n, then what is its common ratio?

(a) m / n (b) n / m
(c) $m + (n / m)$ (d) $n + (m / n)$

(**b**) Let $a, ar, ar^2 \ldots\ldots$ 200 terms be a geometric progression.
Where, a is the first terms and r be the common ratio.
GP of odd terms $a, ar^2, ar^4, \ldots\ldots$ 100 terms.
GP of even terms $ar, ar^3, ar^5, \ldots\ldots$ 100 terms.
$\therefore$ Sum of odd terms of the GP $= m$

$$\Rightarrow \frac{a\{r^{200} - 1\}}{r - 1} = m \quad \ldots\text{(i)}$$

Sum of even terms of the GP $= n$

$$\Rightarrow \frac{ar(r^{200} - 1\}}{r - 1} = n \quad \ldots\text{(ii)}$$

Dividing of Eq. (i) by Eq. (ii),

$$\Rightarrow \frac{1}{r} = \frac{m}{n} \Rightarrow r = \frac{n}{m}$$

Hence, the common ratio of the GP is $\frac{n}{m}$.

9. If a set A contains 3 elements and another set B contains 6 elements, then what is the minimum number of elements that $(A \cup B)$ can have?

(a) 3 (b) 6
(c) 8 (d) 9

(**b**) $n(A) = 3$, $n(B) = 6$
$\therefore$ The minimum number of elements in
$A \cup B = 6$
i.e $n(A \cup B) = 6$
(because max $n(A \cap B) = 3$

10. What is the number of diagonals of an octagon?

(a) 48
(b) 40
(c) 28
(d) 20

(**d**) The number of vertices of an octagon $= 8$
$\therefore$ The number of points in a plane $= 8$
$\therefore$ Total number of straight line form by 8 points $= {}^8C_2$
[$\because$ 1 straight line form by 2 points]

$$= \frac{8!}{2!6!} = \frac{8 \times 7}{2} = 28$$

$\therefore$ The number of diagonals of an octagon = Total number of straight line form by 8 points – number of sides of octagon
$= 28 - 8 = 20$

11. What is the value of the determinant $\begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix}$?

(a) 0 (b) 12
(c) 24 (d) 36

(**c**) Given determinant

$$= \begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix} = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 6 & 24 \\ 6 & 24 & 120 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & 6 \\ 6 & 12 & 48 \end{vmatrix}$$

[by $C_2 \to C_2 - 2C_1, C_3 \to C_3 - 3C_2$]
$= 1(96 - 72) - 0 + 0$
[expression w.r.t. first row]
$= 24$

12. What are the values of x that satisfy the equation

$$\begin{vmatrix} x & 0 & 2 \\ 2x & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 3x & 0 & 2 \\ x^2 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0 ?$$

(a) $-2 \pm \sqrt{3}$
(b) $-1 \pm \sqrt{3}$
(c) $-1 \pm \sqrt{6}$
(d) $-2 \pm \sqrt{6}$

(**d**) Given equation,

$$\begin{vmatrix} x & 0 & 2 \\ 2x & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 3x & 0 & 2 \\ x^2 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$\Rightarrow x(2 - 1) - 0 + 2(2x - 2) + 3x(2 - 1)$
$- 0 + 2\,(x^2 - 0) = 0$
[expression w.r.t. first row]
$\Rightarrow x + 4x - 4 + 3x + 2x^2 = 0$
$\Rightarrow 2x^2 + 8x - 4 = 0$
$\Rightarrow x^2 + 4x - 2 = 0$

$$\Rightarrow x = \frac{-4 \pm \sqrt{16 - 4\,(1)\,(-2)}}{2}$$
$$= \frac{-4 \pm \sqrt{24}}{2} = \frac{-4 \pm 2\sqrt{6}}{2}$$
$$= -2 \pm \sqrt{6}$$

13. If $x + a + b + c = 0$, then what is the value of $\begin{vmatrix} x + a & b & c \\ a & x + b & c \\ a & b & x + c \end{vmatrix}$?

(a) 0 (b) $(a + b + c)^2$
(c) $a^2 + b^2 + c^2$ (d) $a + b + c - 2$

(**a**) Given, $x + a + b + c = 0$

$$\begin{vmatrix} x + a & b & c \\ a & x + b & c \\ a & b & x + c \end{vmatrix}$$
$$= \begin{vmatrix} x + a + b + c & b & c \\ x + a + b + c & x + b & c \\ x + a + b + c & b & x + c \end{vmatrix}$$

[by $C_1 \to C_1 + C_2 + C_3$]

$$= (x + a + b + c) \begin{vmatrix} 1 & b & c \\ 1 & x + b & c \\ 1 & b & x + c \end{vmatrix}$$

[$x + a + b + c$ common from C_1] $= 0$
[$\because x + a + b + c = 0$]

14. If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then the expression $A^3 - 2A^2$ is

(a) a null matrix (b) an identity matrix
(c) equal to A (d) equal to $-A$

(**a**) $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 + 1 & -1 - 1 \\ -1 - 1 & 1 + 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

and $A^3 = A^2 \cdot A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 + 2 & -2 - 2 \\ -2 - 2 & 2 + 2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

Now,

$$A^3 - 2A^2 = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} - 2\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 & -4+4 \\ -4+4 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

= a null matrix

15. Let m and $n (m < n)$ be the roots of the equation $x^2 - 16x + 39 = 0$. If four terms p, q, r and s are inserted between m and n to form an AP, then what is the value of $p + q + r + s$?

(a) 29 (b) 30
(c) 32 (d) 35

(**c**) m and n be the roots of the equation $x^2 - 16x + 39 = 0$ $(m < n)$.

$\therefore$ $m + n = 16$ ...(i)

and $mn = 39$ ...(ii)

We know that, $n - m = \sqrt{(m+n)^2 - 4mn}$ $(\because m < n)$

$= \sqrt{256 - 156} = \sqrt{100}$

$n - m = 10$ ...(iii)

Solving the Eqs. (ii) and (iii), $n = 13, m = 3$

Four terms p, q, r and s are inserted between m and n to form an AP.

$\therefore$ AP is 3, p, q, r, s, 13

Here, $a = 3$, $l = 13$, $n = 6$

$\therefore$ $l = a + (n-1)d$

$13 = 3 + (6-1)d$

$\Rightarrow$ $d = 2$

$\therefore$ $p = a + d = 3 + 2 = 5,$

$q = a + 2d = 3 + 4 = 7$

$r = a + 3d = 3 + 6 = 9,$

$d = a + 4d = 3 + 8 = 11$

Now, $p + q + r + s = 5 + 7 + 9 + 11 = 32$

16. Under which one of the following conditions will the quadratic equation $x^2 + mx + 2 = 0$ always have real roots?

(a) $2\sqrt{3} \le m^2 < 8$ (b) $\sqrt{3} \le m^2 < 4$
(c) $m^2 \ge 8$ (d) $m^2 \le \sqrt{3}$

(**c**) The quadratic equation $x^2 + mx + 2 = 0$, have real roots.

$\therefore$ $m^2 - 4(1)(2) \ge 0$

[quadratic equation $ax^2 + bx + c = 0$ have real roots if $b^2 - 4ac \ge 0$]

$\Rightarrow$ $m^2 - 8 \ge 0$

$\Rightarrow$ $m^2 \ge 8$

17. What is the value of $\left[\dfrac{i+\sqrt{3}}{2}\right]^{2019} + \left[\dfrac{i-\sqrt{3}}{2}\right]^{2019}$?

(a) 1
(b) – 1
(c) $2i$
(d) $-2i$

(**c**) $\left[\dfrac{i+\sqrt{3}}{2}\right]^{2019} + \left[\dfrac{i-\sqrt{3}}{2}\right]^{2019}$

$$= \left[\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]^{2019} - \left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]^{2019}$$

$$= \left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]^{2019} - \left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]^{2019}$$

$$= \cos\frac{2019\pi}{6} + i\sin\frac{2019\pi}{6} - \cos\frac{2019\pi}{6} + i\sin\frac{2019\pi}{6}$$

[De-moivre's theorem $(\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$]

$$= 2i\sin\frac{2019\pi}{6}$$

$$= 2i\sin\left(168 \times 2\pi + \frac{3\pi}{6}\right)$$

$$= 2i\sin\frac{3\pi}{6}$$

[$\because \sin(2n\pi + \theta) = \sin\theta$, n is an integer]

$$= 2i\sin\frac{\pi}{2} = 2i$$

18. If α and β are the roots of $x^2 + x + 1 = 0$, then what is $\sum_{j=0}^{3}(\alpha^j + \beta^j)$ equal to?

(a) 8 (b) 6
(c) 4 (d) 2

(**d**) α and β are the roots of the equation $x^2 + x + 1 = 0$

$\therefore$ $\alpha + \beta = -1$

and $\alpha\beta = 1$

Now, $\sum_{j=0}^{3}(\alpha^j + \beta^j) = (\alpha^0 + \beta^0)$
$+ (\alpha^1 + \beta^1) + (\alpha^2 + \beta^2) + (\alpha^3 + \beta^3)$

$= (1 + 1) + (-1) + \{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta\} + (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$

$= 2 - 1 + \{(\alpha+\beta)^2 - 2\alpha\beta\} + (-1)\{\alpha^2 + \beta^2 + 2\alpha\beta - 3\alpha\beta\}$

$= 1 + \{(-1)^2 - 2(1)\} - \{(\alpha+\beta)^2 - 3(1)\}$

$= 1 - 1 - \{(-1)^2 - 3\}$

$= -(1 - 3) = 2$

19. In a school, 50% students play cricket and 40% play football. If 10% of students play both the games, then what per cent of students play neither cricket nor football?

(a) 10% (b) 15% (c) 20% (d) 25%

(**c**) Students, who play cricket = 50%

Students, who play football = 40%

Students who play both games = 10%

Students who play only cricket = 50 – 10 = 40%

Students who play only football = 40 – 10 = 30%

$\therefore$ Total students who play any game = 40 + 30 + 10 = 80%

$\therefore$ Students who play neither cricket nor football = 100 – 80 = 20%

20. If $A = \{x : 0 \le x \le 2\}$ and $B = \{y : y$ is a prime number$\}$, then what is $A \cap B$ equal to?

(a) ϕ (b) {1} (c) {2} (d) {1, 2}

(**c**) $A = \{x : 0 \le x \le 2\} = \{0, 1, 2\}$

and $B = \{y : y$ is a prime number$\}$ $= \{2, 3, 5, 7, 11, \ldots\}$

$\therefore A \cap B = \{0, 1, 2\} \cap \{2, 3, 5, 7, 11, \ldots\} = \{2\}$

21. If $x = 1 + i$, then what is the value of $x^6 + x^4 + x^2 + 1$?

(a) $6i - 3$ (b) $-6i + 3$
(c) $-6i - 3$ (d) $6i + 3$

(**c**) Given, $x = 1 + i$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Now, $x^6 + x^4 + x^2 + 1$

$= x^4(x^2 + 1) + 1(x^2 + 1)$

$= (x^2 + 1)(x^4 + 1)$

$$= \left[(\sqrt{2})^2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2 + 1\right]\left[(\sqrt{2})^4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^4 + 1\right]$$

$$= \left[2\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right) + 1\right]\left[4\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right) + 1\right]$$

[$\because (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$]

$$= \left[2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) + 1\right][4(\cos\pi + i\sin\pi) + 1]$$

$= [2(0 + i) + 1][4(-1 + 0) + 1]$

$= (2i + 1)(-4 + 1) = -6i - 3$

22. What is the value of $2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots\infty}}}$?

(a) $\sqrt{2}-1$ (b) $\sqrt{2}+1$ (c) 3 (d) 4

(*b*) Let, $x=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots\infty}}}$

$\Rightarrow \quad x=2+\frac{1}{x} \Rightarrow x^2=2x+1$

$\Rightarrow \quad x^2-2x-1=0$

$\Rightarrow \quad x=\frac{2\pm\sqrt{(-2)^2-4(1)(-1)}}{2}$

$=\frac{2\pm\sqrt{8}}{2}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$

$=\sqrt{2}+1 \quad (\because x>2)$

23. If $P(n,r)=2520$ and $C(n,r)=21$, then what is the value of $C(n+1,r+1)$?

(a) 7 (b) 14
(c) 28 (d) 56

(*c*) If $P(n,r)=2520$ and $C(n,r)=21$,

$\therefore \quad {}^nP_r=2520$

$\Rightarrow \quad \frac{n!}{(n-r)!}=2520 \quad \ldots(i)$

and $\quad {}^nC_r=21$

$\Rightarrow \quad \frac{n!}{r!(n-r)!}=21 \quad \ldots(ii)$

From Eqs. (i) and (ii), we get

$\frac{2520}{r!}=21$

$\Rightarrow \quad r!=\frac{2520}{21}=120$

$\Rightarrow \quad r!=5!$

$\therefore \quad r=5$

Putting the value of r in Eq. (i),

$\frac{n!}{(n-5)!}=2520$

$\Rightarrow \quad n(n-1)(n-2)(n-3)(n-4)$

$=7\times6\times5\times4\times3$

$\therefore \quad n=7$

Now, $C(n+1,r+1)={}^{n+1}C_{r+1}$

$={}^{7+1}C_{5+1}={}^8C_6$

$=\frac{8!}{6!2!}=\frac{8\times7}{2}$

$=28$

24. How many terms are there in the expansion of $(1+2x+x^2)^5+(1+4y+4y^2)^5$?

(a) 12 (b) 20
(c) 21 (d) 22

(*d*) Given expansion,

$(1+2x+x^2)^5+(1+4y+4y^2)^5$

$=[(1+x)^2]^5+[(1+2y)^2]^5$

$=(1+x)^{10}+(1+2y)^{10}$

$\therefore$ Total number of terms in given expansion.

$=(10+1)+(10+1)=22$

[$\because$ total number of terms in expansion of $(1+x)^n=n+1$]

25. If the middle term in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$ is $184756x^{10}$, then what is the value of n?

(a) 10 (b) 8
(c) 5 (d) 4

(*a*) The middle term in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$

$=\left(\frac{2n}{2}+1\right)$th term $\quad [\because 2n$ is even$]$

$=(n+1)$th term.

According to the question,

Value of middle term $=184756x^{10}$

$\Rightarrow \quad {}^{2n}C_n(x^2)^{2n-n}\left(\frac{1}{x}\right)^n=184756x^{10}$

[$\because T_{r+1}={}^nC_r\,x^{n-r}a^r$ in expansion of $(x+a)^n$]

$\Rightarrow {}^{2n}C_n(x)^{4n-2n-n}=184756x^{10}$

$\Rightarrow \quad {}^{2n}C_n(x)^n=184756x^{10}$

Comparing the power of x both sides

$n=10$

26. If $A=\begin{pmatrix}1&2\\2&3\\3&4\end{pmatrix}$ and $B=\begin{pmatrix}1&2\\2&1\end{pmatrix}$, then which one of the following is correct?

(a) Both AB and BA exist
(b) Neither AB nor BA exists
(c) AB exists but BA does not exist
(d) AB does not exist but BA exists

(*c*) We have, $A=\begin{pmatrix}1&2\\2&3\\3&4\end{pmatrix}$ and $B=\begin{pmatrix}1&2\\2&1\end{pmatrix}$

order of $A=3\times2$ and order of $B=2\times2$

$\because$ Number of column of A = Number of row of B

$\therefore AB$ exists.

and number of column of $B\neq$ Number of raw of A

$\therefore BA$ does not exist.

Hence, AB exists but BA does not exist.

27. If $n!$ has 17 zeros, then what is the value of n?

(a) 95 (b) 85
(c) 80
(d) No such value of n exists

(*b*) We know that each interval of 5! is one zero.

i.e. 5! has one zero.

10! has two zeros.

$\therefore$ 85! has 17 zeros.

Hence, the value of n is 85.

28. Let $A\cup B=\{x\,|(x-a)(x-b)>0$, where $a<b\}$. What are A and B equal to?

(a) $A=\{x|\,x>a\}$ and $B=\{x|\,x>b\}$
(b) $A=\{x|\,x<a\}$ and $B=\{x|\,x>b\}$
(c) $A=\{x|\,x<a\}$ and $B=\{x|\,x<b\}$
(d) $A=\{x|\,x>a\}$ and $B=\{x|\,x<b\}$

(*c*) Let $A\cup B=\{x:(x-a)(x-b)>0$, where $a<b\}$.

It is possible if $x-a<0$ and $x-b<0$

or $x<a$ and $x<b$

$\therefore A=\{x:x<a\}$ and $B=\{x:x<b\}$

29. If the constant term in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is 405, then what can be the values of k?

(a) ±2 (b) ±3
(c) ±5 (d) ±9

(*b*) Let $(r+1)$th term in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is constant.

$\therefore \quad T_{r+1}={}^{10}C_r(\sqrt{x})^{10-r}\left(\frac{-k}{x^2}\right)^r$

[$\because T_{r+1}={}^nC_rx^{n-r}a^r$ in expansion of $(x+a)^n$]

$\Rightarrow 405={}^{10}C_r\,(x)^{\frac{10-r}{2}-2r}\cdot(-k)^r$

$\Rightarrow 405={}^{10}C_r\,(x)^{\frac{10-5r}{2}}\cdot(-k)^r \quad \ldots(i)$

For constant term

$\frac{10-5r}{2}=0 \Rightarrow 10-5r=0$

$\therefore \quad r=2$

Putting the value of r, in Eq. (i),

$405={}^{10}C_2.(-k)^2$

$\Rightarrow \quad 405=\frac{10!}{2!8!}\times k^2$

$\Rightarrow \quad 405=\frac{10\times9}{2}.k^2$

$\Rightarrow \quad k^2=\frac{405}{45}$

$\Rightarrow \quad k^2=9$

$\Rightarrow \quad k=\pm3$

30. What is $C(47, 4) + C(51, 3) + C(50, 3) + C(49, 3) + C(48, 3) + C(47, 3)$ equal to?

(a) $C(47, 4)$ (b) $C(52, 5)$
(c) $C(52, 4)$ (d) $C(47, 5)$

(**c**) $C(47, 4) + C(51, 3) + C(50, 3) + C(49, 3) + C(48, 3) + C(47, 3)$

$= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3 + {}^{47}C_4$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{48}C_4$

$[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{49}C_4$

$= {}^{51}C_3 + {}^{50}C_3 + {}^{50}C_4$

$= {}^{51}C_3 + {}^{51}C_4$

$= {}^{52}C_4 = C(52, 4)$

31. Let a, b, c be in AP and $k \neq 0$ be a real number. Which of the following are correct?

1. ka, kb, kc are in AP
2. $k - a, k - b, k - c$ are in AP
3. $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are in AP

Select the correct answer using the code given below.

(a) 1 and 2 only (b) 2 and 3 only
(c) 1 and 3 only (d) 1, 2 and 3

(**d**) a, b, c are in AP.

We know that equal number addition, subtraction and multiply, divide, by equal number of each term of an AP, the resultent, series be an AP.

$\therefore ka, kb, kc$ are in AP (multiplying by k).

$k - a, k - b, k - c$ are in AP (subtraction from k) and $\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$ are in AP (divide by k)

Hence, option (d) is correct answer.

32. How many two-digit numbers are divisible by 4?

(a) 21 (b) 22
(c) 24 (d) 25

(**b**) Series of two-digit number that divisible by 4 is

12, 16, 20, ..., 96

This series is an AP

Here, $A = 12, d = 4, l = 96$

Let total number of terms be n.

$\therefore \quad l = a + (n-1)d$

$\Rightarrow \quad 96 = 12 + (n-1)4$

$\Rightarrow \quad 84 = (n-1)4$

$\Rightarrow \quad n - 1 = 21$

$\Rightarrow \quad n = 21 + 1 = 22$

33. Let S_n be the sum of the first n terms of an AP. If $S_{2n} = 3n + 14n^2$, then what is the common difference?

(a) 5 (b) 6
(c) 7 (d) 9

(**c**) $S_{2n} = 3n + 14n^2$ (S_n be the sum of first n terms of an AP)

$\Rightarrow \quad S_{2n} = \frac{3}{2}.(2n) + \frac{7}{2}(2n)^2$

Put $2n = n$

we get, $S_n = \frac{3n}{2} + \frac{7n^2}{2}$

$\therefore \quad T_n = S_n - S_{n-1}$

$= \frac{3}{2}n + \frac{7n^2}{2} - \frac{3}{2}(n-1) - \frac{7}{2}(n-1)^2$

$= \frac{3}{2}n + \frac{7}{2}n^2 - \frac{3}{2}n + \frac{3}{2} - \frac{7}{2}n^2 - \frac{7}{2} + \frac{7}{2}.2n$

$T_n = 7n - 2$

Put $n = 1, 2, ...$

$T_1 = 7(1) - 2 = 5$

$T_2 = 7(2) - 2 = 12$

$\therefore \quad d = T_2 - T_1 = 12 - 5 = 7$

34. If 3rd, 8th and 13th terms of a GP are p, q and r respectively, then which one of the following is correct?

(a) $q^2 = pr$ (b) $r^2 = pq$
(c) $pqr = 1$ (d) $2q = p + r$

(**a**) Let first term and common ratio of a GP be a and R.

$\therefore \quad T_3 = aR^2 = p \quad ...(i)$

$T_8 = aR^7 = q \quad ...(ii)$

$T_{13} = aR^{12} = r \quad ...(iii)$

Multiplying of Eqs. (i) and (iii)

$(aR^2)(aR^{12}) = pr$

$\Rightarrow \quad a^2R^{14} = pr$

$\Rightarrow \quad (aR^7)^2 = pr$

$\Rightarrow \quad q^2 = pr$ [from Eq. (ii)]

35. What is the solution of $x \leq 4, y \geq 0$ and $x \leq -4, y \leq 0$?

(a) $x \geq -4, y \leq 0$ (b) $x \leq 4, y \geq 0$
(c) $x \leq -4, y = 0$ (d) $x \geq -4, y = 0$

(**c**) Given inequalities

$x \leq 4, y \geq 0 \quad ...(i)$

and $\quad x \leq -4, y \leq 0 \quad ...(ii)$

Possible value of x and y.

$x = \{4, 3, 2, 1, 0, -1, -2, -3, -4, -5, ...\}$

$y = \{0, 1, 2, 3, 4 .....\} \quad ...(i)$

and $x = \{-4, -5, -6, -7, ...\}$,

$y = \{0, -1, -2, -3, -4 ...\} \quad ...(ii)$

Take combine (i) and (ii),

$x = \{-4, -5, -6, -7 ... \}, y = 0$

or $\quad x \leq -4, y = 0.$

36. If $x^{\log_7 x} > 7$ where $x > 0$, then which one of the following is correct?

(a) $x \in (0, \infty)$ (b) $x \in \left(\frac{1}{7}, 7\right)$

(c) $x \in \left(0, \frac{1}{7}\right) \cup (7, \infty)$

(d) $x \in \left(\frac{1}{7}, \infty\right)$

(**b**) $x^{\log_7 x} > 7$ where $x > 0$.

Taking log on base 7 both sides

$\log_7 x . \log_7 x > \log_7 7$

$[\because \log_a m^n = n \log_a m]$

$\Rightarrow \quad (\log_7 x)^2 > 1 \quad [\because \log_a a = 1]$

$\Rightarrow \quad \log_7 x > (\pm 1)$

$\therefore \quad x > 7^1 \Rightarrow x > 7$

and $\quad x < 7^{-1} \Rightarrow x < \frac{1}{7}$

Hence, $x \in \left(\frac{1}{7}, 7\right)$

37. How many real roots does the equation $x^2 + 3|x| + 2 = 0$ have?

(a) Zero (b) One
(c) Two (d) Four

(**a**) Given equation, $x^2 + 3|x| + 2 = 0$

Case I $x^2 + 3x + 2 = 0$ (when $x > 0$)

$\Rightarrow \quad x^2 + x + 2x + 2 = 0$

$\Rightarrow \quad x(x+1) + 2(x+1) = 0$

$\Rightarrow \quad (x+1)(x+2) = 0$

$\therefore \quad x = -1, -2$

Hence, no real roots because $x > 0$.

Case II $x^2 - 3x + 2 = 0$ (when $x < 0$)

$\Rightarrow \quad x^2 - 2x - x + 2 = 0$

$\Rightarrow \quad x(x-2) - 1(x-2) = 0$

$\Rightarrow \quad (x-2)(x-1) = 0$

$\therefore \quad x = 1, 2$

Hence, no real roots because $x < 0$.

$\therefore$ The number of real roots of given equation is zero.

38. Consider the following statements in respect of the quadratic equation $4(x - p)(x - q) - r^2 = 0$, where p, q and r are real numbers.

1. The roots are real.
2. The roots are equal, if $p = q$ and $r = 0$.

Which of the above statements is/are correct?

(a) Only 1 (b) Only 2
(c) Both 1 and 2 (d) Neither 1 nor 2

(**c**) Given quadratic equation,

$4(x - p)(x - q) - r^2 = 0$

$\Rightarrow \quad 4x^2 - (4q + 4p)x + 4pq - r^2 = 0$

Comparing it Eq. by $ax^2 + bx + c = 0$

$a = 4, b = -4(p+q), c = 4pq - r^2$

$b^2 - 4ac = 16(p+q)^2 - 4\times 4(4pq - r^2)$

$= 16p^2 + 16q^2 + 32pq - 64pq + 16r^2$

$= 16p^2 + 16q^2 - 32pq + 16r^2$

$= 16(p-q)^2 + 16r^2$

$\because$ $b^2 - 4ac$ will be positive.

So, the roots are real.

If $p = q$ and $r = 0$, then $b^2 - 4ac = 0$

So, the roots are equal.

Hence, the statements both 1 and 2 are correct.

39. Let $S = \{2, 4, 6, 8, ..., 20\}$.

What are the maximum number of subsets of S?

(a) 10 (b) 20
(c) 512 (d) 1024

(*d*) $S = \{2, 4, 6, 8, ..., 20\}$

Here, number of elements of set

$S = 10\,(n)$

$\therefore$ Maximum number of subsets of set

$S = 2^n = 2^{10} = 1024$

40. A binary number is represented by $(cdccddcccddd)_2$, where $c > d$. What is its decimal equivalent?

(a) 1848 (b) 2048
(c) 2842 (d) 2872

(*d*) Binary number $= (cdccddcccddd)_2$ where, $c > d$. We know that only two bit (digits) 0 and 1 be any binary number.

$\therefore$ Given binary number

$= (101100111000)_2$

$= (1\times 2^{11} + 0\times 2^{10} + 1\times 2^9 + 1\times 2^8 +$

$+ 0\times 2^7 + 0\times 2^6 + 1\times 2^5 + 1\times 2^4$

$+ 1\times 2^3 + 0\times 2^2 + 0\times 2^1 + 0\times 2^0)_{10}$

$= (2048 + 512 + 256 + 32 + 16 + 8)_{10}$

$= (2872)_{10}$

41. If $\operatorname{cosec}\theta = \frac{29}{21}$, where $0 < \theta < 90°$, then what is the value of $4\sec\theta + 4\tan\theta$?

(a) 5 (b) 10 (c) 15 (d) 20

(*b*) Given, $\operatorname{cosec}\theta = \frac{29}{21}$

where, $0 < \theta < 90°$

$\because \operatorname{cosec}\theta = \frac{H}{P} = \frac{29}{21} = k$ (let)

$\therefore H = 29k, P = 21k$

$\therefore B = \sqrt{(H)^2 - (P)^2} = \sqrt{(29k)^2 - (21k)^2}$

$= \sqrt{841k^2 - 441k^2}$

$= \sqrt{400k^2} = 20k$

$\therefore \sec\theta = \frac{H}{B} = \frac{29k}{20k} = \frac{29}{20}$

and $\tan\theta = \frac{P}{B} = \frac{21k}{20k} = \frac{21}{20}$

Now, $4\sec\theta + 4\tan\theta$

$= 4\times\frac{29}{20} + 4\times\frac{21}{20}$

$= 4\times\frac{50}{20} = 10$

42. Consider the following statements

1. $\cos\theta + \sec\theta$ can never be equal to 1.5.
2. $\tan\theta + \cot\theta$ can never be less than 2.

Which of the above statements is/are correct?

(a) Only 1 (b) Only 2
(c) Both 1 and 2 (d) Neither 1 nor 2

(*b*) We know that, $-1 \le \cos\theta \le 1$ and $-1 \le \sec\theta \le \infty$ but $\cos\theta = \sec\theta$ if $\theta = 0$ and $\theta = 180°$

$\therefore \quad -2 \le \cos\theta + \sec\theta \le \infty$

So, $\cos\theta + \sec\theta = 1.5$ is possible.

and again $0 \le \tan\theta \le \infty$ and

$0 \le \cot\theta \le \infty$, but $\tan\theta = \cot\theta$

if $\quad \theta = 45°$

$\therefore \quad 2 \le \tan\theta + \cot\theta \le \infty$

So, $\tan\theta + \cot\theta$ can never be less than 2.

Hence, only the Statement 2 is correct.

43. A ladder 9 m long reaches a point 9 m below the top of a vertical flagstaff. From the foot of the ladder, the elevation of the flagstaff is 60°. What is the height of the flagstaff?

(a) 9 m (b) 10.5 m
(c) 13.5 m (d) 15 m

(*) Let AP be a ladder and QR be a vertical flagstaff. P is a point 9 m below the top on flagstaff. A is the foot of ladder and h is the height of point P from the ground.

$\therefore AP = 9$ m, $PR = 9$ m, $PQ = h$m

In $\Delta\, AQP$, $\quad \sin\theta = \frac{PQ}{AP}$

$\Rightarrow \sin 60° = \frac{h}{9} \Rightarrow \frac{\sqrt{3}}{2} = \frac{h}{9}$

$\Rightarrow \quad 9\sqrt{3} = 2h \Rightarrow h = \frac{9\sqrt{3}}{2}$

$= \frac{9\times 1.73}{2} = \frac{15.57}{2} = 7.7$ m

$\therefore$ Height of flagstaff

$= h + 9 = 7.7 + 9$

$= 16.7$ m

44. What is the length of the chord of a unit circle which subtends an angle θ at the centre?

(a) $\sin\left(\frac{\theta}{2}\right)$ (b) $\cos\left(\frac{\theta}{2}\right)$
(c) $2\sin\left(\frac{\theta}{2}\right)$ (d) $2\cos\left(\frac{\theta}{2}\right)$

(*c*) Given, radius of circle = 1 unit

Angle subtends at the centre of circle by chord $= \theta$

We know that, length of chord

$= 2r\sin\frac{\theta}{2} = 2\times 1\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2}$

45. What is $\tan\left\{2\tan^{-1}\left(\frac{1}{3}\right)\right\}$ equal to?

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{3}{8}$
(d) $\frac{1}{9}$

(*b*) $\tan\left\{2\tan^{-1}\left(\frac{1}{3}\right)\right\}$

$= \tan\left\{\tan^{-1}\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right\}$

$\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$

$= \tan\tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right) = \frac{2\times 9}{3\times 8} = \frac{3}{4}$

46. What is the scalar projection of $\mathbf{a} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ on $\mathbf{b} = 4\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$?

(a) $\frac{\sqrt{6}}{9}$ (b) $\frac{19}{9}$ (c) $\frac{9}{19}$ (d) $\frac{\sqrt{6}}{19}$

(*b*) $\mathbf{a} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = 4\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$

Projection of $\mathbf{a}$ on $\mathbf{b}$

$= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

$= \frac{(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}).(4\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}})}{\sqrt{16 + 16 + 49}}$

$= \frac{4 + 8 + 7}{\sqrt{81}} = \frac{19}{9}$

47. If the magnitude of the sum of two non-zero vectors is equal to the magnitude of their difference, then which one of the following is correct?

(a) The vectors are parallel
(b) The vectors are perpendicular
(c) The vectors are anti-parallel
(d) The vectors must be unit vectors

(*b*) Let $\mathbf{a}$ and $\mathbf{b}$ are the two non-zero vectors.

According to the question,

$|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

$\Rightarrow \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}\cdot\mathbf{b}$

$\Rightarrow \quad 4\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow \mathbf{a}\cdot\mathbf{b} = 0$

So, $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

48. Consider the following equations for two vectors **a** and **b**.

1. $(\mathbf{a}+\mathbf{b}).(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$
2. $(|\mathbf{a}+\mathbf{b}|)(|\mathbf{a}-\mathbf{b}|) = |\mathbf{a}|^2 - |\mathbf{b}|^2$
3. $|\mathbf{a}\cdot\mathbf{b}| + |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}^2|\,|\mathbf{b}|^2$

Which of the above statements are correct?

(a) 1, 2 and 3 (b) Only 1 and 2
(c) Only 1 and 3 (d) Only 2 and 3

(*c*) I : $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$

$= \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}$

$|\mathbf{a}|^2 - \mathbf{a}\,.\,\mathbf{b} + \mathbf{a}\,.\,\mathbf{b} - |\mathbf{b}|^2$ $[\because \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}]$

$= |\mathbf{a}|^2 - |\mathbf{b}|^2$

So, Statement 1 is correct.

2. $(|\mathbf{a}+\mathbf{b}|)(|\mathbf{a}-\mathbf{b}|) = |\mathbf{a}+\mathbf{b}||\mathbf{a}-\mathbf{b}| \neq |\mathbf{a}|^2 - |\mathbf{b}|^2$

So, Statement 2 is not correct.

3. $|\mathbf{a}\cdot\mathbf{b}|^2 + |\mathbf{a}\times\mathbf{b}|^2 = ||\mathbf{a}||\mathbf{b}|\cos\theta|^2 + ||\mathbf{a}||\mathbf{b}|\sin\theta|^2$

$= |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta + |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta$

$= |\mathbf{a}|^2|\mathbf{b}|^2(\cos^2\theta + \sin^2\theta)$

$= |\mathbf{a}|^2|\mathbf{b}|^2$ $[\because \cos^2\theta + \sin^2\theta = 1]$

So, statement 3 is correct.

Hence, only Statements 1 and 3 are correct.

49. Consider the following statements.

1. The magnitude of $\mathbf{a}\times\mathbf{b}$ is same as the area of a triangle with sides **a** and **b**
2. If $\mathbf{a}\times\mathbf{b} = 0$, where $\mathbf{a}\neq 0, \mathbf{b}\neq 0$, then $\mathbf{a} = \lambda\,\mathbf{b}$.

Which of the above statements is/are correct?

(a) Only 1 (b) Only 2
(c) Both 1 and 2 (d) Neither 1 nor 2

(*b*) 1. We know that,

Area of triangle with sides **a** and **b**

$= \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

So, statement 1 is not correct.

2. $\mathbf{a}\times\mathbf{b} = 0$, where $\mathbf{a}\neq 0$, $\mathbf{b} = 0$,

So, **a** and **b** are parallel.

$\Rightarrow \mathbf{a} = \lambda\,\mathbf{b}$

So, Statement 2 is correct.

Hence, only statement 2 is correct.

50. If **a** and **b** are unit vectors and θ is the angle between them, then what is $\sin^2\left(\frac{\theta}{2}\right)$ equal to?

(a) $\frac{|\mathbf{a}+\mathbf{b}|^2}{4}$ (b) $\frac{|\mathbf{a}-\mathbf{b}|^2}{4}$

(c) $\frac{|\mathbf{a}+\mathbf{b}|^2}{2}$ (d) $\frac{|\mathbf{a}-\mathbf{b}|^2}{2}$

(*b*) Given, $|\mathbf{a}| = 1, |\mathbf{b}| = 1$

We know that,

$|\mathbf{a}-\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b}$

$\Rightarrow |\mathbf{a}-\mathbf{b}|^2 = 1 + 1 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$

$\Rightarrow |\mathbf{a}-\mathbf{b}|^2 = 2 - 2\cos\theta$

$\Rightarrow |\mathbf{a}-\mathbf{b}|^2 = 2(1-\cos\theta)$

$\Rightarrow |\mathbf{a}-\mathbf{b}|^2 = 2\left(1 - 1 + 2\sin^2\frac{\theta}{2}\right)$

$\Rightarrow |\mathbf{a}-\mathbf{b}|^2 = 2\cdot\left(2\sin^2\frac{\theta}{2}\right)$

$\Rightarrow \sin^2\frac{\theta}{2} = \frac{|\mathbf{a}-\mathbf{b}|^2}{4}$

51. The equation $ax + by + c = 0$ represents a straight line

(a) for all real numbers, *a*, *b* and *c*
(b) only when $a \neq 0$
(c) only when $b \neq 0$
(d) only when at least one of *a* and *b* is non-zero.

(*d*) The equation $ax + by + c = 0$ represents a straight line only when at least one of *a* and *b* is non zero.

52. What is the angle between the lines $x\cos\alpha + y\sin\alpha = a$ and $x\sin\beta - y\cos\beta = a$?

(a) $\beta - \alpha$ (b) $\pi + \beta - \alpha$
(c) $\frac{(\pi + 2\beta + 2\alpha)}{2}$ (d) $\frac{(\pi - 2\beta + 2\alpha)}{2}$

(*d*) The equations of given lines

$x\cos\alpha + y\sin\alpha = a$ ... (i)

and $x\sin\beta - y\cos\beta = a$ ... (ii)

Slope of Eq. (i), $m_1 = \frac{-\cos\alpha}{\sin\alpha} = -\cot\alpha$

$= \tan\left(\frac{\pi}{2} + \alpha\right)$

Slope of Eq. (ii), $m_2 = \frac{-\sin\beta}{-\cos\beta} = \tan\beta$

Let θ be the angle between the lines, then

$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

$= \frac{\tan\left(\frac{\pi}{2}+\alpha\right) - \tan\beta}{1 + \tan(\pi - \alpha)\tan\beta}$

$\tan\theta = \tan\left(\frac{\pi}{2} + \alpha - \beta\right)$

$\theta = \frac{\pi}{2} + \alpha - \beta = \frac{\pi + 2\alpha - 2\beta}{2}$

53. What is the distance between the points $P(m\cos 2\alpha, m\sin 2\alpha)$ and $Q(m\cos 2\beta, m\sin 2\beta)$?

(a) $|2m\sin(\alpha-\beta)|$ (b) $|2m\cos(\alpha-\beta)|$
(c) $|m\sin(2\alpha - 2\beta)|$
(d) $|m\sin(2\alpha - 2\beta)|$

(*a*) Given points, $p(m\cos 2\alpha, m\sin 2\alpha)$ and $Q(m\cos 2\beta, m\sin 2\beta)$

$\therefore PQ = \left|\sqrt{(m\cos 2\beta - m\cos 2\alpha)^2 + (m\sin 2\beta - m\sin 2\alpha)^2}\right|$

[$\because$ Distance between two points (x_1, y_1) and (x_2, y_2)

$= |\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}|$]

$= \left|\sqrt{m^2\cos^2 2\beta + m^2\cos^2 2\alpha - 2m^2\cos 2\beta\cos 2\alpha + m^2\sin^2 2\beta + m^2\sin^2 2\alpha - 2m^2\sin 2\beta\sin 2\alpha}\right|$

$= \left|m\sqrt{(\cos^2 2\beta + \sin^2 2\beta) + (\cos^2 2\alpha + \sin^2 2\alpha) - 2(\cos 2\beta\cos 2\alpha + \sin 2\beta\sin 2\alpha)}\right|$

$= |m\sqrt{1 + 1 - 2\cos(2\alpha - 2\beta)}|$

$= |m\sqrt{2[1 - \cos 2(\alpha - \beta)]}|$

$= \left|m\sqrt{2\left[1 - 1 + 2\sin^2\frac{2(\alpha-\beta)}{2}\right]}\right|$

$= |m\sqrt{2\times 2\sin^2(\alpha-\beta)}|$

$= |2m\sin(\alpha - \beta)|$

54. An equilateral triangle has one vertex at $(-1, -1)$ and another vertex at $(-\sqrt{3}, \sqrt{3})$. The third vertex may lie on

(a) $(-\sqrt{2}, \sqrt{2})$ (b) $(\sqrt{2}, -\sqrt{2})$
(c) (1, 1) (d) (1, – 1)

(*c*) Consider two vertices of an equilateral triangle are $A(-1, -1)$ and $B(-\sqrt{3}, \sqrt{3})$. Let third vertex *x* be $C(x, y)$.

$\because \Delta ABC$ is equilateral

$\therefore AC = AB \sqrt{(x+1)^2 + (y+1)^2}$

$= \sqrt{(-\sqrt{3}+1)^2 + (\sqrt{3}+1)^2}$

$\Rightarrow x^2 + 1 + 2x + y^2 + 1 + 2y$

$= 3 + 1 - 2\sqrt{3} + 3 + 1 + 2\sqrt{3}$

$\Rightarrow x^2 + y^2 + 2x + 2y + 2 = 8$

$\Rightarrow x^2 + y^2 + 2x + 2y = 6$

From option only point (1, 1) is satisfying of it equation. Hence, the third vertex may lie on (1, 1).

55. If the angle between the lines joining the end points of minor axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with one of the its foci is $\frac{\pi}{2}$, then what is the eccentricity of the ellipse?

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2\sqrt{2}}$

(*b*) Equation of the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

End points of minor axis are $(0, b)$, $(0, -b)$ and one foci is $(ae, 0)$

Slope of line $BS = \frac{0 - b}{ae - 0} = -\frac{b}{ae}(m_1)$

Slope of line $B'S = \frac{0 + b}{ae - 0} = \frac{b}{ae}(m_2)$

According to the question, angle between BS and $B'S$ is $\frac{\pi}{2}$.

i.e. BS and $B'S$ are perpendicular,

$\therefore \quad m_1 m_2 = -1$

$$\frac{-b}{ae} \times \frac{b}{ae} = -1 \Rightarrow b^2 = a^2 e^2 \quad \dots \text{(i)}$$

We know that, $e = \sqrt{1 - \frac{b^2}{a^2}}$

$\Rightarrow \quad a^2 e^2 = a^2 - b^2$

$\Rightarrow \quad a^2 e^2 = a^2 - a^2 e^2$ [from Eq. (i)]

$\Rightarrow \quad 2a^2 e^2 = a^2 \Rightarrow e^2 = \frac{1}{2}$

$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$

56. A point on a line has coordinates $(p+1, p-3, \sqrt{2}p)$ where p is any real number. What are the direction cosines of the line?

(a) $\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$

(c) $\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2}$

(d) Cannot be determined due to insufficient data

(***d***) Coordinate of a point on a line is $(p+1, p-3, \sqrt{2}p)$, p is any real number.

Equation of a line, whose direction ratios are a, b and c and passing through the point (x_1, y_1, z_1)

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r$$

$\therefore$ $(ar + x_1, br + y_1, cr + z_1)$ any point on the line.

According to the questions,

$(ar + x_1, br + y_1, cr + z_1)$

$$= (p+1, p-3, \sqrt{2}p)$$

$\therefore \quad ar = p + 1 - x_1 \quad \dots \text{(i)}$

$br = p - 3 - y_1 \quad \dots \text{(ii)}$

$cr = \sqrt{2}p - z_1 \quad \dots \text{(iii)}$

Squaring and adding of (i), (ii) and (iii)

$(a^2 + b^2 + c^2)r^2 = (p + 1 - x_1)^2$

$+ (p - 3 - y_1)^2 + (\sqrt{2}p - z_1)^2$

We can not find the values of a, b and c.

Hence, the direction cosines of the line can not be determined due to insufficient data.

57. A point on the line

$$\frac{x-1}{1} = \frac{y-3}{2} = \frac{z+2}{7}$$

has coordinates

(a) (3, 5, 4) (b) (2, 5, 5)

(c) (– 1, – 1, 5) (d) (2, – 1, 0)

(***b***) Equation of the line

$$\frac{x-1}{1} = \frac{y-3}{2} = \frac{z+2}{7}$$

From option, point (2, 5, 5) is satisfying the given equation of line.

$$\left[\because \frac{2-1}{1} = \frac{5-3}{2} = \frac{5+2}{7} \Rightarrow 1 = 1 = 1\right]$$

Hence, the coordinates of required point (2, 5, 5).

58. If the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies on the plane $2x - 4y + z = 7$, then what is the value of k?

(a) 2 (b) 3

(c) 5 (d) 7

(***d***) Equation of line

$$= \frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2} = r$$

$\therefore$ $(r + 4, r + 2, 2r + k)$ point lies on the line.

This line lies on the plane

$$2x - 4y + z = 7$$

Then, the point $(r + 4, r + 2, 2r + k)$ lies on the plane, we get

$$2(r + 4) - 4(r + 2) + (2r + k) = 7$$

$\Rightarrow \quad 2r + 8 - 4r - 8 + 2r + k = 7$

$\Rightarrow \quad k = 7$

Hence, the value of k is 7.

59. A straight line passes through the point (1, 1, 1) makes an angle 60° with the positive direction of Z-axis, and the cosine of the angles made by it with the positive directions of the Y-axis and the X-axis are in the ratio $\sqrt{3} : 1$. What is the acute angle between the two possible positions of the line?

(a) 90° (b) 60°

(c) 45° (d) 30°

(***b***) Let the straight line makes the angle with X-axis, Y-axis and Z-axis be α, β and γ.

$\therefore \gamma = 60°$ and $\frac{\cos\beta}{\cos\alpha} = \frac{\sqrt{3}}{1}$

If l, m and n are the direction cosines of the lines, then

$$n = \cos\gamma = \cos 60° = \frac{1}{2}$$

and $\frac{m}{l} = \frac{\cos\beta}{\cos\alpha} = \frac{\sqrt{3}}{1}$

$\Rightarrow \quad \frac{m}{l} = \frac{\sqrt{3}}{1} = k$ (Let)

$\therefore \quad m = \sqrt{3}k, l = k$

We know that, $l^2 + m^2 + n^2 = 1$

$k^2 + 3k^2 + \frac{1}{4} = 1$

$\Rightarrow \quad 4k^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\Rightarrow \quad k^2 = \frac{3}{16} \Rightarrow k = \pm\frac{\sqrt{3}}{4}$

$\therefore \quad l_1 = \frac{\sqrt{3}}{4}, m_1 = \frac{3}{4}, n_1 = \frac{1}{2}$

and $l_2 = -\frac{\sqrt{3}}{4}, m_2 = -\frac{3}{4}, n_2 = \frac{1}{2}$

We know that,

$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

$\Rightarrow \cos\theta = \left|-\frac{3}{16} - \frac{9}{16} + \frac{1}{4}\right|$

$$= \left|\frac{-3 - 9 + 4}{16}\right| = \left|\frac{-8}{16}\right|$$

$\cos\theta = \frac{1}{2} = \cos 60°$

$\therefore \quad \theta = 60°$

60. If the points $(x, y, -3)$, $(2, 0, -1)$ and $C(4, 2, 3)$ lie on a straight line, then what are the values of x and y respectively?

(a) 1, – 1 (b) –1, 1

(c) 0, 2 (d) 3, 4

(***a***) Points, $A(x, y, -3)$, $B(2, 0, -1)$ and (4, 2, 3). These points lie on a straight line, then direction ratios of $AB = \lambda$ (direction ratios of BC)

$\therefore (2 - x, 0 - y, -1 + 3)$

$$= (4 - 2, 2 - 0, 3 + 1)$$

$\Rightarrow (2 - x, -y, 2) = (2, 2, 4)$

$\Rightarrow (2 - x, -y, 2) = 2(1, 1, 2)$

Comparing both sides,

$2 - x = 1 \Rightarrow x = 1$

and $-y = 1 \Rightarrow y = -1$

61. What is the minimum value of $\frac{a^2}{\cos^2 x} + \frac{b^2}{\sin^2 x}$ where $a > 0$ and $b > 0$?

(a) $(a + b)^2$ (b) $(a - b)^2$

(c) $a^2 + b^2$ (d) $|a^2 + b^2|$

(*) Let $p = \frac{a^2}{\cos^2 x} + \frac{b^2}{\sin^2 x}$

$$= a^2\sec^2 x + b^2\text{cosec}^2 x$$

$$- 2ab\sec x \,\text{cosec}\, x$$

$$+ 2ab\sec x \,\text{cosec}\, x$$

$$= (a\sec x - b\,\text{cosec}\, x)^2$$

$$+ 2ab\sec x\,\text{cosec}\, x$$

For minimum value of p,

$a\sec x - b\,\text{cosec}\, x = 0$

$\Rightarrow \quad a\sec x = b\,\text{cosec}\, x$

$\Rightarrow \quad \frac{\sec x}{\text{cosec}\, x} = \frac{b}{a}$

$\Rightarrow \quad \tan x = \frac{b}{a}$

$\therefore \sin x = \frac{b}{\sqrt{a^2 + b^2}}, \cos x = \frac{a}{\sqrt{a^2 + b^2}}$

$\therefore$ Minimum value of p

$$= \frac{a^2(a^2 + b^2)}{a^2} + \frac{b^2(a^2 + b^2)}{b^2}$$

$$= 2(a^2 + b^2)$$

62. If the angles of a triangle ABC are in AP and $b : c = \sqrt{3} : \sqrt{2}$, then what is the measure of angle A?

(a) 30° (b) 45°
(c) 60° (d) 75°

(*d*) Angles of a triangle ABC are in AP, then $2B = A + C$

We know that, $A + B + C = 180°$

$\Rightarrow 3B = 180° \Rightarrow B = 60°$

By sine rule, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\Rightarrow \frac{\sin A}{a} = \frac{\sin 60°}{b} = \frac{\sin C}{c}$

Take II and III, $\frac{\sin 60°}{b} = \frac{\sin C}{c}$

$\Rightarrow \frac{\sin 60°}{\sin C} = \frac{b}{c} \Rightarrow \frac{\sqrt{3}/2}{\sin C} = \frac{\sqrt{3}}{\sqrt{2}}$

[$\because$ Given, $b : c = \sqrt{3} : \sqrt{2}$]

$\Rightarrow \sin C = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin C = \sin 45° \Rightarrow C = 45°$

$\therefore A = 180° - (B + C)$

$= 180° - (60° + 45°) = 75°$

63. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then what is the value of $\cot (A - B)$?

(a) $\frac{1}{x} + \frac{1}{y}$ (b) $\frac{1}{y} - \frac{1}{x}$
(c) $\frac{xy}{x + y}$ (d) $1 + \frac{1}{xy}$

(*a*) Given, $\tan A - \tan B = x$ ... (i)

and $\cot B - \cot A = y$ ... (ii)

From Eq. (i), $\tan A - \tan B = x$

$\Rightarrow \frac{1}{\cot A} - \frac{1}{\cot B} = x$

$\Rightarrow \frac{\cot B - \cot A}{\cot A \cot B} = x$

$\Rightarrow \cot A \cot B = \frac{y}{x}$ [from Eq. (ii)]

Now, $\cot (A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$

$= \frac{\frac{y}{x} + 1}{y} = \frac{y + x}{xy} = \frac{1}{x} + \frac{1}{y}$

64. What is $\sin (\alpha + \beta) - 2 \sin \alpha \cos \beta + \sin (\alpha - \beta)$ equal to?

(a) 0 (b) $2 \sin \alpha$
(c) $2 \sin \beta$ (d) $\sin \alpha + \sin \beta$

(*a*) $\sin (\alpha + \beta) - 2 \sin \alpha \cos \beta + \sin (\alpha - \beta)$

$= \sin \alpha \cos \beta + \cos \alpha \sin \beta - 2 \sin \alpha \cos \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta = 0$

65. If $2 \tan A = 3 \tan B = 1$, then what is $\tan (A - B)$ equal to?

(a) $\frac{1}{5}$ (b) $\frac{1}{6}$
(c) $\frac{1}{7}$ (d) $\frac{1}{9}$

(*c*) Given, $2 \tan A = 3 \tan B = 1$

$\therefore \tan A = \frac{1}{2}, \tan B = \frac{1}{3}$

Now, $\tan (A - B)$

$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$= \frac{\frac{1}{2} - \frac{1}{3}}{1 + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{\frac{3-2}{6}}{\frac{6+1}{6}} = \frac{1}{7}$

66. What is $\cos 80° + \cos 40° - \cos 20°$ equal to?

(a) 2 (b) 1
(c) 0 (d) –19

(*c*) $\cos 80° + \cos 40° - \cos 20°$

$= 2 \cos \frac{80° + 40°}{2} . \cos \frac{80° - 40°}{2} - \cos 20°$

$= 2 \cos 60° \cos 20° - \cos 20°$

$= 2 \times \frac{1}{2} \cos 20° - \cos 20°$

$= \cos 20° - \cos 20° = 0$

67. If angle C of a triangle ABC is a right angle, then what is $\tan A + \tan B$ equal to ?

(a) $\frac{a^2 - b^2}{ab}$ (b) $\frac{a^2}{bc}$
(c) $\frac{b^2}{ca}$ (d) $\frac{c^2}{ab}$

(*d*) In ΔABC, $\angle C = 90°$

$\therefore c^2 = a^2 + b^2$

[by Pythagoras theorem] ... (i)

$\tan A = \frac{a}{b}, \tan B = \frac{b}{a}$

Now, $\tan A + \tan B = \frac{a}{b} + \frac{b}{a}$

$= \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}$ [from Eq. (i)]

68. What is $\cot \left(\frac{A}{2}\right) - \tan \left(\frac{A}{2}\right)$ equal to?

(a) $\tan A$ (b) $\cot A$
(c) $2 \tan A$ (d) $2 \cot A$

(*d*) $\cot \frac{A}{2} - \tan \frac{A}{2}$

$= \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} - \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}$

$= \frac{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}{\sin \frac{A}{2} \cos \frac{A}{2}} = \frac{2 \cos A}{2 \sin \frac{A}{2} \cos \frac{A}{2}}$

$= \frac{2 \cos A}{\sin A} = 2 \cot A$

69. What is $\cot A + \text{cosec } A$ equal to?

(a) $\tan \left(\frac{A}{2}\right)$ (b) $\cot \left(\frac{A}{2}\right)$
(c) $2 \tan \left(\frac{A}{2}\right)$ (d) $2 \cot \left(\frac{A}{2}\right)$

(*b*) $\cot A + \text{cosec } A$

$= \frac{\cos A}{\sin A} + \frac{1}{\sin A} = \frac{\cos A + 1}{\sin A}$

$= \frac{2 \cos^2 \frac{A}{2} - 1 + 1}{2 \sin \frac{A}{2} \cos \frac{A}{2}}$

$= \frac{2 \cos^2 \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}} = \cot \left(\frac{A}{2}\right)$

70. What is $\tan 25° \tan 15° + \tan 15° \tan 50° + \tan 25° \tan 50°$ equal to?

(a) 0 (b) 1
(c) 2 (d) 4

(*b*) $\because \tan 50° = \tan (90° - 40°)$

$\Rightarrow \tan 50° = \cot 40°$

$\Rightarrow \tan 50° = \frac{1}{\tan 40°}$

$\Rightarrow \tan 50° = \frac{1}{\tan (25° + 15°)}$

$\Rightarrow \tan 50° = \frac{1 - \tan 25° \tan 15°}{\tan 25° + \tan 15°}$

$\Rightarrow \tan 25° \tan 50° + \tan 15° \tan 50° = 1 - \tan 25° \tan 15°$

$\Rightarrow \tan 25° \tan 15° + \tan 15° \tan 50° + \tan 25° \tan 50° = 1$

71. What is the area of the region bounded by $|x| < 5$, $y = 0$ and $y = 8$?

(a) 40 sq units (b) 80 sq units
(c) 120 sq units (d) 160 sq units

(*b*) Given curve $y = 0$ and $y = 8$ and $|x| < 5$

Case I When $x < 0$, then

area of the region bounded

$= \int_{-5}^{0} 0 \, dx - \int_{-5}^{0} 8 \, dx = 0 - 8 [x]_{-5}^{0}$

$= - 8 [0 + 5] = - 40$

$= 40$ sq units

[$\because$ area will not be negative]

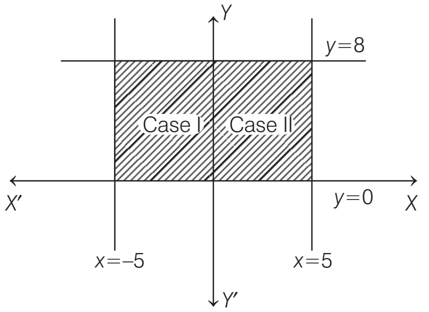
Case II when $x > 0$, then

Area of the region bounded

$= \int_{0}^{5} 0 \, dx - \int_{0}^{5} 8 \, dx = 0 - 8 [x]_{0}^{5}$

$= - 8[5 - 0] = - 40 = 40$ sq units

$\therefore$ Required area = 40 + 40 = 80 sq units



72. Consider the following statements in respect of the function $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$:

1. $\lim_{x \to 0} f(x)$ exists
2. $f(x)$ is continuous at $x = 0$

Which of the above statement is/are correct?

(a) Only 1 (b) Only 2
(c) Both 1 and 2 (d) Neither 1 nor 2

(*d*) Given, $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\text{LHL} = \lim_{x \to 0^-} \sin\left(\frac{1}{x}\right) = \lim_{h \to 0} \sin\left(\frac{1}{0 - h}\right)$$

$$= \lim_{h \to 0} -\sin\left(\frac{1}{h}\right) = -\sin\infty$$

$$= -\text{(a rational number)}$$

$[\because \sin\theta$ lies between –1 to 1]

$$\text{RHL} = \lim_{x \to 0^+} \sin\left(\frac{1}{x}\right) = \lim_{h \to 0} \sin\left(\frac{1}{0 + h}\right)$$

$$= \lim_{h \to 0} \sin\left(\frac{1}{h}\right) = \sin\infty$$

= a rational number

$[\because \sin\theta$ lies between – 1 to 1]

$\because$ LHL $\neq$ RHL

So, $f(x)$ does not exists.

$\because f(x) = 0$ at $x = 0$

$\therefore$ LHL $\neq$ RHL $\neq f(0)$

So, $f(x)$ is not continuous.

Hence, the statements neither 1 nor 2 correct.

73. What is the value of $\lim_{x \to 0} \frac{\sin x^\circ}{\tan 3x^\circ}$?

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 1

(*b*) $$\lim_{x \to 0} \frac{\sin x^\circ}{\tan 3x^\circ} = \lim_{x \to 0} \left(\frac{x \times \frac{\sin x}{x}}{3x \times \frac{\tan 3x}{3x}} \right)$$

$$= \frac{1}{3} \lim_{x \to 0} \frac{\left(\frac{\sin x}{x}\right)}{\left(\frac{\tan 3x}{3x}\right)} = \frac{1}{3}$$

$$\left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \text{ and } \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1\right]$$

74. What is the degree of the differential equation

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 - x^2\left(\frac{d^4y}{dx^4}\right) = 0?$$

(a) 1 (b) 2
(c) 3 (d) 4

(***a***) Given differential equation,

$$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 - x^2\left(\frac{d^4y}{dx^4}\right) = 0$$

$$\Rightarrow \frac{d^4y}{dx^4} - \frac{1}{x^2}\left(\frac{d^3y}{dx^3}\right) - \frac{1}{x^2}\left(\frac{dy}{dx}\right)^2 = 0$$

We know that power of the heighest order of differentiation is the degree of differential equation.

So, the degree of it equation is 1.

75. Which one of the following is the second degree polynomial function $f(x)$ where, $f(0) = 5$, $f(-1) = 10$ and $f(1) = 6$?

(a) $5x^2 - 2x + 5$ (b) $3x^2 - 2x - 5$
(c) $3x^2 - 2x + 5$ (d) $3x^2 - 10x + 5$

(***c***) From the option (c),

$$f(x) = 3x^2 - 2x + 5$$

$$f(0) = 3(0)^2 - 2(0) + 5$$

$$= 5$$

$$f(-1) = 3(-1)^2 - 2(-1) + 5$$

$$= 3 + 2 + 5 = 10$$

and $f(1) = 3(1)^2 - 2(1) + 5$

$$= 3 - 2 + 5 = 6$$

Hence, the required polynomial

$$f(x) = 3x^2 - 2x + 5.$$

Directions (Q. Nos. 76-78) *Read the following information and answer the three items that follow .*

A curve $y = me^{mx}$ where $m > 0$ intersects Y-axis at a point P.

76. What is the slope of the curve at the point of intersection P?

(a) m (b) m^2
(c) $2m$ (d) $2m^2$

77. How much angle does the tangent at P make with y-axis?

(a) $\tan^{-1} m^2$

(b) $\cot^{-1}(1 + m^2)$

(c) $\sin^{-1}\left(\frac{1}{\sqrt{1 + m^4}}\right)$

(d) $\sec^{-1}\sqrt{1 + m^4}$

78. What is the equation of tangent to the curve at P?

(a) $y = mx + m$ (b) $y = -mx + 2m$
(c) $y = m^2x + 2m$ (d) $y = m^2x + m$

Solutions (Q. Nos. 76-78)

Given curve $y = me^{mx}$ where $m > 0$

$\because$ Curve intersects Y-axis at a point P, then $x = 0$

$$\therefore \quad y = me^0 \Rightarrow y = m$$

$\therefore$ Point P (0, m)

Now, differentiation w.r.t x of given curve,

$$\frac{dy}{dx} = m.e^{mx}.m$$

$$\frac{dy}{dx} = m^2e^{mx}$$

at point $P(0, m)$, $\frac{dy}{dx} = m^2e^0 = m^2$

76. (***b***) Slope of the curve at the point $P(0, m)$

$$= \left(\frac{dy}{dx}\right) \text{at point } P\ (0, m) = m^2$$

77. (***c***) Let the tangent makes the angle with X-axis be θ, then

$$\tan\theta = \left(\frac{dy}{dx}\right) \text{at } P\ (0, m)$$

$$\Rightarrow \quad \tan\theta = m^2 \Rightarrow \theta = \tan^{-1} m^2$$

Now, the tangent will make the angle with Y-axis

$$= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \tan^{-1} m^2$$

$$= \cot^{-1} m^2 \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{1 + m^4}}\right)$$

$$\left[\because \cot^{-1} x = \sin\left(\frac{1}{\sqrt{1 + m^2}}\right)\right]$$

78. (***d***) Equation of tangent to curve at P is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} (x - x_1)$$

$$\Rightarrow y - m = m^2\ (x - 0)$$

$$\Rightarrow \quad y = m^2x + m$$

Directions (Q. Nos. 79 and 80) *Read the following information and answer the two items that follow.*

Let $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log x$.

79. For $x = \frac{\sqrt{\pi}}{2}$, what is the value of $[ho(gof)](x)$?

(a) 0 (b) 1 (b) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

80. What is $[fo(fof)](2)$ equal to ?

(a) 2 (b) 8
(c) 16 (d) 256

Solutions (Q. Nos. 79 and 80) Given, $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log x$

79. (**a**) $(gof)(x) = g\{f(x)\} = \tan x^2$

Now, $[ho(gof)](x) = h\{(gof)(x)\}$
$= \log(\tan x^2)$

for $x = \frac{\sqrt{\pi}}{2}$

$[ho(gof)]\left(\frac{\sqrt{\pi}}{2}\right) = \log \tan\left(\frac{\pi}{4}\right)$
$= \log 1 = 0$

80. (**d**) $(fof)(x) = f\{f(x)\}$
$= (x^2)^2 = x^4$

Now, $[fo(fof)](x) = f\{(fof)(x)\}$
$= (x^4)^2 = x^8$

$\therefore [fo(fof)](2) = 2^8 = 256$

81. What is $\int \frac{dx}{2x^2 - 2x + 1}$ equal to?

(a) $\frac{\tan^{-1}(2x-1)}{2} + c$

(b) $2\tan^{-1}(2x-1) + c$

(c) $\frac{\tan^{-1}(2x+1)}{2} + c$

(d) $\tan^{-1}(2x-1) + c$

(**d**) Let $I = \int \frac{dx}{2x^2 - 2x + 1}$

$$= \frac{1}{2}\int \frac{dx}{x^2 - x + \frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{x^2 - x + \frac{1}{4} - \frac{1}{4} + \frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2} \cdot 2\tan^{-1}\left[\frac{\left(x - \frac{1}{2}\right)}{\frac{1}{2}}\right] + c$$

$$\left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$= \tan^{-1}(2x - 1) + c$

82. What is $\int \frac{dx}{x(1 + \ln x)^n}$ equal to $(n \neq 1)$?

(a) $\frac{1}{(n-1)(1+\ln x)^{n-1}} + c$

(b) $\frac{1-n}{(1+\ln x)^{1-n}} + c$

(c) $\frac{n+1}{(1+\ln x)^{n+1}} + c$

(d) $-\frac{1}{(n-1)(1+\ln x)^{n-1}} + c$

(**d**) Suppose,

$$I = \int \frac{dx}{x(1+\ln x)^n} \quad (\text{where } n \neq 1)$$

Let $1 + \ln x = t$

Diff. w.r.t. x, we get

$$0 + \frac{1}{x} = \frac{dt}{dx} \Rightarrow \frac{dx}{x} = dt$$

$$\therefore \quad I = \int \frac{dt}{t^n} = \frac{t^{-n+1}}{-n+1} + c$$

$$= -\frac{1}{(n-1)t^{n-1}} + c$$

$$= -\frac{1}{(n-1)(1+\log x)^{n-1}} + c$$

83. Which one of the following is the differential equation that represents the family of curves $y = \frac{1}{2x^2 - C}$, where C is an arbitrary constant?

(a) $\frac{dy}{dx} = 4xy^2$ (b) $\frac{dy}{dx} = \frac{1}{y}$

(c) $\frac{dy}{dx} = x^2y$ (d) $\frac{dy}{dx} = -4xy^2$

(**d**) The differential equation of family of curves $y = \frac{1}{2x^2 - C}$ ... (i)

(where, C is any arbitrary constant)

Differentiation w.r.t. x of Eq. (i)

$$\frac{dy}{dx} = -\frac{1}{(2x^2 - C)^2}\frac{d}{dx}(2x^2 - C)$$

$$\Rightarrow \frac{dy}{dx} = -y^2.(4x - 0)$$

$\Rightarrow \frac{dy}{dx} = -4xy^2$, it is required differential equation.

Directions (Q. Nos. 84 and 85) *Read the following information and answer the two items that follow.*

Consider the equation $x^y = e^{x-y}$

84. What is $\frac{dy}{dx}$ at $x = 1$ equal to?

(a) 0 (b) 1
(c) 2 (d) 4

85. What is $\frac{d^2y}{dx^2}$ at $x = 1$ equal to?

(a) 0 (b) 1
(c) 2 (d) 4

Solutions (Q. Nos 84 and 85)

Given equation, $x^y = e^{x-y}$

On taking log both sides, we get

$$y \log x = (x - y)\log e$$

$$\Rightarrow \quad y\log x = x - y \quad [\because \log_e e = 1]$$

$$\Rightarrow (1 + \log x)y = x \Rightarrow y = \frac{x}{(1+\log x)}$$

Differentiation w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x\left(0 + \frac{1}{x}\right)}{(1+\log x)^2}$$

$$\frac{dy}{dx} = \frac{1 + \log x - 1}{(1+\log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

84. (a) $\because \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

At $x = 1$, $\frac{dy}{dx} = \frac{\log 1}{(1+\log 1)^2}$

$= \frac{0}{1} = 0 \quad [\because \log 1 = 0]$

85. (b) $\because \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Differentiation. w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{(1+\log x)^2.\frac{1}{x} - (\log x).2(1+\log x)\left(0 + \frac{1}{x}\right)}{(1+\log x)^4}$$

$$= \frac{\frac{1}{x}(1+\log x)(1+\log x - 2\log x)}{(1+\log x)^4}$$

$$= \frac{\frac{1}{x}(1+\log x)(1-\log x)}{(1+\log x)^4}$$

At $x = 1$ $\frac{d^2y}{dx^2} = \frac{1(1+0)(1-0)}{(1+0)^4} = 1$

Directions (Q.Nos. 86-88) *Read the following information and answer the three items that follow.*

Consider the function

$$f(x) = g(x) + h(x)$$

where, $g(x) = \sin\left(\frac{x}{4}\right)$

and $h(x) = \cos\left(\frac{4x}{5}\right)$

86. What is the period of the function $g(x)$?

(a) π (b) 2π
(c) 4π (d) 8π

87. What is the period of the function $h(x)$?

(a) π (b) $\frac{4\pi}{5}$
(c) $\frac{5\pi}{2}$ (d) $\frac{3\pi}{2}$

88. What is the period of the function $f(x)$?

(a) 10π (b) 20π
(c) 40π (d) 80π

Solutions (Q. Nos. 86-88)

Given, $f(x) = g(x) + h(x)$,

where, $g(x) = \sin\left(\frac{x}{4}\right)$ and

$$h(x) = \cos\left(\frac{4x}{5}\right)$$

86. (*d*) $g(x) = \sin\left(\frac{x}{4}\right)$

$$g(x + 8\pi) = \sin\left(\frac{x+8\pi}{4}\right)$$
$$= \sin\left(2\pi + \frac{x}{4}\right)$$
$$= \sin\left(\frac{x}{4}\right) = g(x)$$

$\therefore$ Period of the function $g(x) = 8\pi$

87. (*c*) $h(x) = \cos\left(\frac{4x}{5}\right)$

$$h\left(x + \frac{5\pi}{2}\right) = \cos\frac{4}{5}\left(x + \frac{5\pi}{2}\right)$$
$$= \cos\left(2\pi + \frac{4x}{5}\right)$$
$$= \cos\left(\frac{4x}{5}\right) = h(x)$$

$\therefore$ Pperiod of the function $h(x) = \frac{5\pi}{2}$

88. (*c*) $f(x) = g(x) + h(x)$

$$= \sin\left(\frac{x}{4}\right) + \cos\left(\frac{4x}{5}\right)$$
$$f(x + 40\pi) = \sin\left(\frac{x+40\pi}{4}\right) + \cos\frac{4}{5}(x + 40\pi)$$
$$= \sin\left(10\pi + \frac{x}{4}\right) + \cos\left(32\pi + \frac{4x}{5}\right)$$
$$= \sin\left(5\times 2\pi + \frac{x}{4}\right) + \cos\left(16\times 2\pi + \frac{4x}{5}\right)$$
$$= \sin\left(\frac{x}{4}\right) + \cos\left(\frac{4x}{5}\right) = f(x)$$

$\therefore$ Period of the function $f(x) = 40\pi$

Directions (Q. Nos. 89 and 90) *Read the following information and answer the two items that follow.*

consider the function

$$f(x) = 3x^4 - 20x^3 - 12x^2 + 288x + 1$$

89. In which one of the following intervals is the function increasing?

(a) (–2, 3) (b) (3, 4)
(c) (–3, – 2) (d) (–4, – 3)

90. In which one of the following intervals is the function decreasing?

(a) (–2, 3) (b) (3, 4)
(c) (4, 6) (d) (6, 9)

Solutions (Q. Nos. 89 and 90)

Given function,

$f(x) = 3x^4 - 20x^3 - 12x^2 + 288x + 1$

Differentiation w.r.t. x, we get

$$f'(x) = 12x^3 - 60x^2 - 24x + 288$$

89. (*b*) $f(x)$ is increasing, if $f'(x) \geq 0$

$$12x^3 - 60x^2 - 24x + 288 \geq 0$$
$$\Rightarrow \quad x^3 - 5x^2 - 2x + 24 \geq 0$$
$$\Rightarrow \quad (x+2)(x^2 - 7x + 12) \geq 0$$
$$\Rightarrow \quad (x+2)(x-3)(x-4) \geq 0$$
$$\therefore \quad x \leq -2,\ x \geq 3,\ x \geq 4$$

Hence, $f(x)$ is increasing the interval (3, 4).

90. (*a*) $f(x)$ is decreasing, if $f'(x) \leq 0$

$$12x^3 - 60x^2 - 24x + 288 \leq 0$$
$$\Rightarrow \quad x^3 - 5x^2 - 2x + 24 \leq 0$$
$$\Rightarrow \quad (x+2)(x^2 - 7x + 12) \leq 0$$
$$\Rightarrow \quad (x+2)(x-3)(x-4) \leq 0$$
$$\therefore \quad x \geq -2,\ x \leq 3,\ x \leq 4$$

Hence $f(x)$ is decreasing the interval (–2, 3).

Directions (Q.Nos. 91-93) *Read the following information and answer the two items that follow .*

Let $f(x) = x^2 + 2x - 5$

and $g(x) = 5x + 30$

91. What are the roots of the equation $g[(f(x)] = 0$?

(a) 1, –1 (b) –1, – 1
(c) 1, 1 (d) 0, 1

92. Consider the following statements.

1. $f[g(x)]$ is a polynomial of degree 3.
2. $g[g(x)]$ is a polynomial of degree 2.

Which of the above statements is/are correct?

(a) Only 1 (b) Only 2
(c) Both 1 and 2 (d) Neither 1 nor 2

93. If $h(x) = 5f(x) - xg(x)$, then what is the derivative of $h(x)$?

(a) –40 (b) –20
(c) –10 (d) 0

Solutions (Q. Nos. 91-93) Given,

$f(x) = x^2 + 2x - 5$, $g(x) = 5x + 30$

$$\therefore g[f(x)] = 5(x^2 + 2x - 5) + 30$$
$$= 5x^2 + 10x + 5$$
$$f[g(x)] = (5x + 30)^2 + 2(5x + 30) - 5$$
$$= 25x^2 + 900 + 300x + 10x + 60 - 5$$
$$= 25x^2 + 310x + 955$$

and $g[g(x)] = 5(5x + 30) + 30$
$$= 25x + 180$$

91. (*b*) The equation, $g[f(x)] = 0$

$$5x^2 + 10x + 5 = 0$$
$$\Rightarrow \quad x^2 + 2x + 1 = 0$$
$$\Rightarrow \quad (x+1)^2 = 0$$
$$\therefore \quad x = -1, -1$$

Hence, the roots of this equation are –1, – 1.

92. (*d*) 1. $f[g(x)] = 25x^2 + 310x + 955$

$f[g(x)]$ is a polynomial of degree 2.

So, Statement 1 is not correct.

2. $g[g(x)] = 25x + 180$

$g[g(x)]$ is a polynomial of degree 1.

So, Statement 2 is not correct.

Hence, the Statement neither 1 nor 2 correct.

93. (*b*) Given, $h(x) = 5f(x) - xg(x)$

$$= 5(x^2 + 2x - 5) - x(5x + 30)$$
$$= 5x^2 + 10x - 25 - 5x^2 - 30x$$
$$= -20x - 25$$

Differentiation w.r.t. x, we get

$$h'(x) = -20$$

Hence, derivative of $h(x)$ is –20.

Directions (Q.Nos. 94 and 95) *Read the following information and answer the questions given below.*

Consider the integrals

$$I_1 = \int_0^\pi \frac{x\,dx}{1 + \sin x} \text{ and}$$
$$I_2 = \int_0^\pi \frac{(\pi - x)dx}{1 - \sin(\pi + x)}$$

94. What is the value of I_1?

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) 2π

95. What is the value of $I_1 + I_2$?

(a) 2π (b) π (c) $\frac{\pi}{2}$ (d) 0

Solutions (Q.Nos. 94 and 95)

Given, $I_1 = \int_0^\pi \frac{x\,dx}{1+\sin x}$,

$$I_2 = \int_0^\pi \frac{(\pi - x)\,dx}{1 - \sin(\pi + x)}$$
$$I_1 = \int_0^\pi \frac{x\,dx}{1+\sin x} \quad \ldots(i)$$
$$= \int_0^\pi \frac{(\pi - x)\,dx}{1 + \sin(\pi - x)}$$
$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$
$$I_1 = \int_0^\pi \frac{(\pi - x)\,dx}{1 + \sin x} \quad \ldots(ii)$$

Adding Eqs. (i) and (ii), we get

$$2I_1 = \int_0^\pi \frac{(x + \pi - x)\,dx}{1 + \sin x} = \int_0^\pi \frac{\pi\,dx}{1 + \sin x}$$
$$= \pi\int_0^\pi \frac{(1 - \sin x)\,dx}{(1 - \sin^2 x)}$$
$$= \pi\int_0^\pi \frac{(1 - \sin x)\,dx}{\cos^2 x}$$
$$= \pi\int_0^\pi (\sec^2 x - \sec x \tan x)\,dx$$
$$= \pi\,[\tan x - \sec x]_0^\pi$$
$$= \pi\,[(\tan\pi - \sec\pi) - (\tan 0 - \sec 0)]$$

$= \pi\ [(0+1)-(0-1)]$

$2I_1 = \pi\,(1+1) = 2\pi$

$\Rightarrow \quad I_1 = \pi$

Now, $I_2 = \int_0^{\pi} \frac{(\pi - x)\,dx}{1-\sin(\pi + x)}$

$= \int_0^{\pi} \frac{(\pi - \pi + x)\,dx}{1-\sin(\pi - \pi - x)}$

$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$= \int_0^{\pi} \frac{x\,dx}{1-\sin(-x)}$

$= \int_0^{\pi} \frac{x\,dx}{1+\sin x} [\because \sin(-\theta) = \sin\theta]$

$= I_1 = \pi$

94. (**c**) $I_1 = \int_0^{\pi} \frac{x\,dx}{1+\sin x} = \pi$

95. (**a**) $I_1 + I_2 = \pi + \pi = 2\pi$

96. The differential equation which represents the family of curves given by $\tan y = C(1-e^x)$ is

(a) $e^x \tan y\,dx + (1-e^x)\,dy = 0$
(b) $e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$
(c) $e^x (1-e^x)\,dx + \tan y\,dy = 0$
(d) $e^x \tan y\,dy + (1-e^x)\,dx = 0$

(**b**) The equation of the family of curves.

$\tan y = C\,(1-e^x)$ … (i)

Differentiation w.r.t. x, we get

$\sec^2 y \cdot \frac{dy}{dx} = C(0-e^x)$

$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} = -Ce^x$ … (ii)

Eleminating C from Eqs. (i) and (ii), we get

$\sec^2 y \cdot \frac{dy}{dx} = -e^x \frac{\tan y}{(1-e^x)}$

$\Rightarrow (1-e^x)\sec^2 y\,dy = -e^x \tan y \cdot dx$

$\Rightarrow e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0$

97. What is the derivative of $2^{(\sin x)^2}$ with respect to $\sin x$?

(a) $\sin x\, 2^{(\sin x)^2} \ln 4$
(b) $2\sin x\, 2^{(\sin x)^2} \ln 4$
(c) $\ln(\sin x)\, 2^{(\sin x)^2}$
(d) $2\sin x \cos x\, 2^{(\sin x)^2}$

(**a**) Derivative of $2^{(\sin x)^2}$ with respect to $\sin x$

$= \frac{\frac{d}{dx} 2^{(\sin x)^2}}{\frac{d}{dx}(\sin x)} = \frac{2^{(\sin x)^2} \log 2 \frac{d}{dx}(\sin x)^2}{\cos x}$

$= \frac{2^{(\sin x)^2} \log 2 .\, 2 \sin x . \cos x}{\cos x}$

$= 2 \log 2 . (\sin x) . 2^{(\sin x)^2}$

$= \sin x . 2^{(\sin x)^2} . \log 4$

98. For what value of k is the function
$f(x) = \begin{cases} 2x + \frac{1}{4}, & x < 0 \\ k, & x = 0 \\ \left(x + \frac{1}{2}\right)^2, & x > 0 \end{cases}$ continuous?

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

(**a**) Given,

$f(x) = \begin{cases} 2x + \frac{1}{4}, & x < 0 \\ k, & x = 0 \\ \left(x + \frac{1}{2}\right)^2, & x > 0 \end{cases}$ continuous.

$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0^-} \left(2x + \frac{1}{4}\right)$

$= \lim_{h \to 0} \left[2(0-h) + \frac{1}{4}\right]$

$= \lim_{h \to 0} \left(-2h + \frac{1}{4}\right) = \frac{1}{4}$

and $f(0) = k$

$\because$ The function $f(x)$ is continuous at $x = 0$

$\therefore \quad \text{LHL} = f(0) = \text{RHL}$

$\Rightarrow \quad \text{LHL} = f(0) \Rightarrow \frac{1}{4} = k$

Hence, $k = \frac{1}{4}$

99. What is the area of the region enclosed between the curve $y^2 = 2x$ and the straight line $y = x$?

(a) $\frac{2}{3}$ sq unit (b) $\frac{4}{3}$ sq unit
(c) $\frac{1}{3}$ sq unit (d) 1 sq unit

(**a**) Equations of curves,

$y^2 = 2x$ … (i)

and $y = x$ … (ii)

From Eqs. (i) and (ii),

$y^2 = 2y \Rightarrow y(y-2) = 0$

$\therefore \quad y = 0, 2$

$\therefore$ Required area $= \int_0^2 \left(\frac{y^2}{2} - y\right) dy$

$= \left[\frac{1}{2} \cdot \frac{y^3}{3} - \frac{y^2}{2}\right]_0^2$

$= \left[\frac{1}{6}(2)^3 - \frac{(2)^2}{2}\right] - 0$

$= \frac{8}{6} - \frac{4}{2} = \frac{8-12}{6}$

$= -\frac{4}{6} = \frac{-2}{3} = \frac{2}{3}$ sq unit

($\therefore$ area will not be negative)

100. If $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 6x + 7$ increases in the interval T and decreases in the interval S, then which one of the following is correct?

(a) $T = (-\infty, 2), \cup (3, \infty)$ and $S = (2, 3)$
(b) $T = \phi$ and $S = (-\infty, \infty)$
(c) $T = (-\infty, \infty)$ and $S = \phi$
(d) $T = (2, 3)$ and $S = (-\infty, 2) \cup (3, \infty)$

(**a**) Given, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 6x + 7$

Differentiating w.r.t x, we get

$f'(x) = \frac{3x^2}{3} - \frac{5}{2} . 2x + 6 = x^2 - 5x + 6$

$\because f(x)$ is increases in interval T,

$\therefore \quad f'(x) \ge 0$

$\Rightarrow \quad x^2 - 5x + 6 \ge 0$

$\Rightarrow \quad x^2 - 3x - 2x + 6 \ge 0$

$\Rightarrow \quad x(x-3) - 2\,(x-3) \ge 0$

$\Rightarrow (x-3)\,(x-2) \ge 0 \Rightarrow x \le 2, x \ge 3$

$\therefore \quad T = (-\infty, 2) \cup (3, \infty)$

Again, $f(x)$ is decreases in interval S.

$\therefore \quad f'(x) \le 0$

$\Rightarrow \quad x^2 - 5x + 6 \le 0$

$\Rightarrow x^2 - 3x - 2x + 6 \le 0$

$\Rightarrow x(x-3) - 2(x-3) \le 0$

$\Rightarrow \quad (x-3)\,(x-2) \le 0$

$\Rightarrow \quad 2 < x < 3 \Rightarrow x \in (2, 3)$

$\therefore \quad S = (2, 3)$

101. A coin is biased so that heads comes up thrice as likely as tails. For three independent tosses of a coin, what is the probability of getting at most two tails?

(a) 0.16 (b) 0.48
(c) 0.58 (d) 0.98

(**d**) Let X be a random variable, it represents of the number tail comes of three tosses of a coin.

$\therefore$ Possible value of X are 0, 1, 2, 3.

According to the question, the coin is biased in which the probability to comes head is thrice as likely as tails.

$\therefore P(\text{H}) = \frac{3}{4}$ and $P(\text{T}) = \frac{1}{4}$

$P(\text{X} = 0) = P(\{\text{HHH}\}) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$

$P(X = 1) = \text{P}$ (2 heads and 1 tail)

$= P(\{\text{HHT}\}) + P(\{\text{HTH}\}) + P(\{\text{THH}\})$

$= \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}$

$= \frac{27}{64}$

$P(X = 2) = P$ (1 head and 2 tails)

$= P(\{\text{HTT}\}) + P(\{\text{THT}\}) + P(\{\text{TTH}\})$

$= \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}$

$= \frac{9}{64}$

∴ Required probability

$= P(X = 0) + P(X = 1) + P(X = 2)$

$= \frac{27}{64} + \frac{9}{64} + \frac{27}{64}$

$= \frac{63}{64} = 0.98$

102. A bag contains 20 books out of which 5 are defective. If 3 of the books are selected at random and removed from the bag in succession without replacement, then what is the probability that all three books are defective?

(a) 0.009 (b) 0.016
(c) 0.026 (d) 0.047

(***a***) Total books in bag = 20

Defective books = 5

∴ Undefective books = 20 − 5 = 15

∴ Probability to selected three books are defective without replacement

$= \frac{5}{20} \times \frac{4}{19} \times \frac{3}{18}$

$= \frac{6}{684}$

$= 0.0087 = 0.009$

103. The median of the observations 22, 24, 33, 37, $x + 1$, $x + 3$, 46, 47, 57, 58 in ascending order is 42. What are the values of 5th and 6th observations respectively?

(a) 42, 45 (b) 41, 43
(c) 43, 46 (d) 40, 40

(***b***) The observations in ascending order are

22, 24, 33, 37, $x + 1$, $x + 3$, 46, 47, 57, 58

Here, $n = 10$

∴ Median

$$= \frac{\text{Value of } \frac{N}{2} \text{ the observations} + \text{Values of } \left(\frac{N}{2} + 1\right)\text{th observations}}{2}$$

$$\Rightarrow 42 = \frac{\text{Value of 5th observations} + \text{Value of 6th observation}}{2}$$

$\Rightarrow 84 = x + 1 + x + 3$

$\Rightarrow 2x = 84 - 4$

$\Rightarrow x = \frac{80}{2} = 40$

∴ 5th observation $= x + 1 = 40 + 1 = 41$

and 6th observation

$= x + 3 = 40 + 3 = 43$

104. Arithmetic mean of 10 observations is 60 and sum of squares of deviations from 50 is 5000. What is the standard deviation of the observations?

(a) 20 (b) 21
(c) 22.36 (d) 24.70

(***a***) Arithmetic mean of 10 observations = 60

$\therefore \Sigma x_i = 60 \times 10 = 600 \quad \left[\because \bar{x} = \frac{\Sigma x_i}{n}\right]$

If, $A = 50$, then $\Sigma d_i^2 = 5000$

$\because d_i = x_i - A$

$\therefore \Sigma d_i = \Sigma(x_i - A) = \Sigma x_i - A\Sigma 1$

$= 600 - 50 \times n \quad [\because \Sigma 1 = n]$

$= 600 - 50 \times 10 = 100$

Now, $\text{SD} = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$

$= \sqrt{\frac{5000}{10} - \left(\frac{100}{10}\right)^2}$

$= \sqrt{500 - 100} = \sqrt{400} = 20$

105. If p and q are the roots of the equation $x^2 - 30x + 221 = 0$, what is the value of $p^3 + q^3$?

(a) 7010 (b) 7110
(c) 7210 (d) 7240

(***b***) Since, p and q are the roots of the equation

$x^2 - 30x + 221 = 0$

$\therefore p + q = 30$ and $pq = 221$

Now, $p^3 + q^3 = (p + q)(p^2 + q^2 - pq)$

$= 30[p^2 + q^2 + 2pq - 3pq]$

$= 30[(p + q)^2 - 3pq]$

$= 30[(30)^2 - 663]$

$= 30[900 - 663]$

$= 30 \times 237 = 7110$

106. For the variables x and y, the two regression lines are $6x + y = 30$ and $3x + 2y = 25$. What are the values of $\bar{x}$, $\bar{y}$ and r respectively?

(a) $\frac{20}{3}, \frac{35}{9}, -0.5$ (b) $\frac{20}{3}, \frac{35}{9}, 0.5$
(c) $\frac{35}{9}, \frac{20}{3}, -0.5$ (d) $\frac{35}{9}, \frac{20}{3}, 0.5$

(***c***) Given lines, $6x + y = 30$ …(i)

and $3x + 2y = 25$ …(ii)

where, x and y are two variables.

Solving these equations,

$x = \frac{35}{9}$, and $y = \frac{20}{3}$

These lines are regression,

Then, $\bar{x} = \frac{35}{9}, \bar{y} = \frac{20}{3}$

and $r = -\frac{3}{6}$ or $-\frac{1}{2}$

$= -0.5$

107. The class marks in a frequency table are given to be 5, 10, 15, 20, 25, 30, 35, 40, 45, 50. The class limits of the first five classes are

(a) 3-7, 7-13, 13-17, 17-23, 23-27
(b) 2.5-7.5, 7.5-12.5, 12.5-17.5, 17.5-22.5, 22.5-27.5
(c) 1.5-8.5, 8.5-11.5, 11.5,-18.5, 18.5-21.5, 21.5-28.5
(d) 2-8, 8-12, 12-18, 18-22, 22-28

(***b***) Given, class marks in a frequency table are

5, 10, 15, 20, 25, 30, 35, 40, 45, 50.

Let L_1 and L_2 be the lower limit and upper limit of first interval.

$\because$ Class mark $= \frac{L_1 + L_2}{2}$ $5 = \frac{L_1 + L_2}{2}$

$\Rightarrow L_1 + L_2 = 10$ … (i)

and $L_2 - L_1$ = Class interval

or $L_2 - L_1 = 5$ … (ii)

Solving Eq. (i) and (ii),

$L_2 = 7.5$ and $L_1 = 2.5$

∴ Class limit of first classes is 2.5 – 7.5

Similarly find class limit of other classes.

Hence, class limits of the first five classes are

2.5 – 7.5, 7.5 – 12.5, 12.5 – 17.5,
17.5 – 22.5, 22.5 –27.5.

108. The mean of 5 observations is 4.4 and variance is 8.24. If three of the five observations are 1, 2 and 6, then what are the other two observations?

(a) 9, 16 (b) 9, 4
(c) 81, 16 (d) 81, 4

(***b***) Let x_1, x_2, x_3, x_4 and x_5 are five observations.

$\therefore x_1 = 1, x_2 = 2$, and $x_3 = 6$

$\therefore \bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$

$\Rightarrow 4.4 = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}$

$\Rightarrow x_1 + x_2 + x_3 + x_4 + x_5 = 22$

$\Rightarrow 1 + 2 + 6 + x_4 + x_5 = 22$

$\Rightarrow x_4 + x_5 = 22 - 9$

$\Rightarrow x_4 + x_5 = 13$ …(i)

and variance,

$$\sigma^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + (x_4 - \bar{x})^2 + (x_5 - \bar{x})^2}{5}$$

$$\Rightarrow 8.24 = \frac{(1 - 4.4)^2 + (2 - 4.4)^2 + (6 - 4.4)^2 + (x_4 - 4.4)^2 + (x_5 - 4.4)^2}{5}$$

$\Rightarrow 8.24 \times 5 = 11.56 + 5.76 + 2.56 + (x_4 - 4.4)^2 + (13 - x_4 - 4.4)^2$ [from Eq. (i)]

$\Rightarrow 41.20 = 19.88 + (x_4 - 4.4)^2 + (8.6 - x_4)^2$

$\Rightarrow 41.20 - 19.88 = x_4^2 + 19.36 - 8.8x_4 + 73.96 + x_4^2 - 17.2x$

$\Rightarrow 21.32 = 2x_4^2 - 26x_4 + 93.32$

$\Rightarrow 2x_4^2 - 26x_4 + 72 = 0$

$\Rightarrow x_4^2 - 13x_4 + 36 = 0$

$\Rightarrow x_4^2 - 9x_4 - 4x_4 + 36 = 0$

$\Rightarrow x_4(x_4 - 9) - 4(x_4 - 9) = 0$

$\Rightarrow (x_4 - 9)(x_4 - 4) = 0$

$\therefore x_4 = 4, 9$

From Eq. (i), $x_5 = 9, 4$

Hence, other two observations are 9 and 4.

109. If a coin is tossed till the first head appears, then what will be the sample space?

(a) {H} (b) {TH}
(c) {T, HT, HHT, HHHT, ...}
(d) {H, TH, TTH, TTTH, ...}

(**a**) A coin is tossed till the first head appears, then the sample space will be = {H}

110. Consider the following discrete frequency distribution.

x	1	2	3	4	5	6	7	8
f	3	15	45	57	50	36	25	9

What is the value of median of the distribution?

(a) 4 (b) 5 (c) 6 (d) 7

(**b**)

x	f	C
1	3	3
2	15	18
3	45	63
4	57	120
5	50	170
6	36	236
7	25	261
8	9	270
	N = 270	

Here, $N = 270$

$\therefore$ Median

$$= \frac{\text{Value of } \frac{N}{2}\text{th term} + \text{value of } \left(\frac{N}{2} + 1\right)\text{th term}}{2}$$

$$= \frac{\text{Value of 135th term} + \text{Value of 136th term}}{2}$$

$$= \frac{5+5}{2} = 5$$

111. Two dice are thrown simultaneously. What is the probability that the sum of the numbers appearing on them is a prime number?

(a) $\frac{5}{12}$ (b) $\frac{1}{2}$
(c) $\frac{7}{12}$ (d) $\frac{2}{3}$

(**a**) Total number of sample space of two dice are thrown, $n(s) = 6 \times 6 = 36$

Total number of favourable outcomes the sum of numbers appearing on them is a prime number.

(1, 1), (1, 2), (1, 4), (1, 6), (2, 1), (2, 3), (2, 5), (3, 2), (3, 4), (4, 1), (4, 3), (5, 2), (5, 6), (6, 1), (6, 5)

$\therefore n(E) = 15$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{15}{36} = \frac{5}{12}$

112. If 5 of a Company's 10 delivery trucks do not meet emission standards and 3 of them are chosen for inspection, then what is the probability that none of the trucks chosen will meet emission standards?

(a) $\frac{1}{8}$ (b) $\frac{3}{8}$
(c) $\frac{1}{12}$ (d) $\frac{1}{4}$

(**c**) Total trucks of a company's = 10

Number of trucks that do not meet emission standards = 5

Number of trucks that are chosen for inspection = 3

$\therefore$ Required probability $= \frac{^5C_3}{^{10}C_3}$

$$= \frac{\frac{5!}{3!2!}}{\frac{10!}{3!7!}} = \frac{5!7!}{10!2!} = \frac{5 \cdot 4 \cdot 3}{10 \cdot 9 \cdot 8} = \frac{1}{12}$$

113. There are 3 coins in a box. One is a two-headed coin; another is a fair coin; and third is biased coin that comes up heads 75% of time. When one of the three coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

(a) $\frac{2}{9}$ (b) $\frac{1}{3}$
(c) $\frac{4}{9}$ (d) $\frac{5}{9}$

(**c**) Let E_1, E_2 and E_3 represent the events of two-headed coin, a fair coin and biased coin respectively.

$\therefore P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}, P(E_3) = \frac{1}{4}$

$P\left(\frac{E}{E_1}\right) = \frac{1}{2}, P\left(\frac{E}{E_2}\right) = \frac{1}{2}, P\left(\frac{E}{E_3}\right) = \frac{1}{4}$

Apply Baye's theorem,

$$P\left(\frac{E_1}{E}\right) = \frac{P(E_1) \cdot P\left(\frac{E}{E_1}\right)}{P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) + P(E_3) \cdot P\left(\frac{E}{E_3}\right)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4} + \frac{1}{16}}$$

$$= \frac{\frac{1}{4}}{\frac{4+4+1}{16}} = \frac{4}{9}$$

114. Consider the following statements:

1. If A and B are mutually exclusive events, then it is possible that $P(A) = P(B) = 0.6$.
2. If A and B are any two events such that $P(A/B) = 1$, then $P(\bar{B}/\bar{A}) = 1$.

Which of the above statement is/are correct?

(a) Only 1 (b) Only 2
(c) Both 1 and 2 (d) Neither 1 nor 2

(**b**) **Statement 1** : A and B are mutually exclusive events, then $P(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B)$
$= 0.6 + 0.6$
$= 1.2$, it is not possible

So, Statement 1 is not correct.

Statement 2 : A and B are any two events such that

$P\left(\frac{A}{B}\right) = 1$

$\Rightarrow \frac{P(A \cap B)}{P(B)} = 1 \Rightarrow P(A \cap B) = P(B)$ ... (i)

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(B)$ [from Eq. (i)]

$\Rightarrow P(A \cup B) = P(A)$ ... (ii)

Now, $P\left(\frac{\bar{B}}{\bar{A}}\right) = \frac{P(\bar{B} \cap \bar{A})}{P(\bar{A})} = \frac{P(\overline{A \cup B})}{P(\bar{A})}$

$= \frac{1 - P(A \cup B)}{1 - P(A)} = \frac{1 - P(A)}{1 - P(A)} = 1$

So, Statement 2 is correct.

Hence, only the Statement 2 is correct.

115. If a fair die is rolled 4 times, then what is the probability that there are exactly 2 sixes?

(a) $\frac{5}{216}$ (b) $\frac{25}{216}$

(c) $\frac{125}{216}$ (d) $\frac{175}{216}$

(*b*) Let X be a random variable that represents to appearing 6 of rolled a die.

Probability of to get 6 to rolled a die,

$$p = \frac{1}{6}$$

$\therefore$ Probability of not get 6 to rolled a die,

$$q = 1 - \frac{1}{p} = 1 - \frac{1}{6} = \frac{5}{6}$$

Here, $n = 4$, $r = 2$

$\therefore$ Required probability $= {}^nC_r p^r q^{n-r}$

[By Bernoulli distribution]

$$= {}^4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$$

$$= \frac{4!}{2!2!} \times \frac{1}{36} \times \frac{25}{36}$$

$$= \frac{4 \cdot 3}{2 \cdot 1} \times \frac{1}{36} \times \frac{25}{36} = \frac{25}{216}$$

116. Mean of 100 observations is 50 and standard deviation is 10. If 5 is added to each observation, then what will be the new mean and new standard deviation respectively?

(a) 50, 10 (b) 50, 15

(c) 55, 10 (d) 55, 15

(*c*) Mean of 100 observations = 50

and standard deviation = 10

We know that, if k is added to each observation, then new mean will be more than k and standard deviation no change.

$\therefore$ After 5 added to each observation.

mean = 50 + 5 = 55

and standard deviation = 10

117. If the range of a set of observations on a variable X is known to be 25 and if $Y = 40 + 3X$, then what is the range of the set of corresponding observations on Y?

(a) 25 (b) 40

(c) 75 (d) 115

(*c*) Range of set of observations on a variable, $X = 25$

We know that

Range, $R_X = X_{max} = X_{min}$

$\Rightarrow \quad 25 = X_{max} - 0 \quad [\because X_{min} = 0]$

$\Rightarrow \quad X_{max} = 25$

$\because \quad Y = 40 + 3X$

$\therefore \quad Y_{min} = 40 + 3X_{min}$

$= 40 + 3(0) \quad [\because X_{min} = 0]$

$= 40$

and $Y_{max} = 40 + 3X_{max}$

$= 40 + 3(25) \quad [\because X_{max} = 25]$

$= 40 + 75 = 115$

Now, $R_Y = Y_{max} - Y_{min}$

$= 115 - 40 = 75$

118. If V is the variance and M is the mean of first 15 natural numbers, then what is $V + M^2$ equal to?

(a) $\frac{124}{3}$ (b) $\frac{148}{3}$

(c) $\frac{248}{3}$ (d) $\frac{124}{9}$

(*c*) Mean of first 15 natural numbers, M

$$= \frac{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15}{15}$$

$$= \frac{15(15 + 1)}{2 \times 15}$$

$$\left[\because 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}\right]$$

$= 8$

Variance of first 15 natural numbers, V

$$= \frac{1}{15}[(1-8)^2 + (2-8)^2 + (3-8)^2 + (4-8)^2 + (5-8)^2 + (6-8)^2 + (7-8)^2 + (8-8)^2 + (9-8)^2 + (10-8)^2 + (11-8)^2 + (12-8)^2 + (13-8)^2 + (14-8)^2 + (15-8)^2]$$

$$= \frac{1}{15}[(-7)^2 + (-6)^2 + (-5)^2 + (-4)^2 + (-3)^2 + (-2)^2 + (-1)^2 + 0 + (1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2 + (6)^2 + (7)^2]$$

$$= \frac{2}{15}[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2]$$

$$= \frac{2}{15} \times \frac{7(7+1)(14+1)}{6}$$

$$\left[\because 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{2}{15} \times \frac{7 \times 8 \times 15}{6} = \frac{56}{3}$$

Now, $V + M^2 = \frac{56}{3} + 64$

$$= \frac{56 + 192}{3}$$

$$= \frac{248}{3}$$

119. A car travels first 60 km at a speed of 3 v km/h and travels next 60 km at 2 v km/h. What is the average speed of the car?

(a) 2.5 v km/h

(b) 2.4 v km/h

(c) 2.2 v km/h

(d) 2.1 v km/h

(*b*) Time taken for first 60 km with speed 3 v km/h

$$= \frac{60}{3v} = \frac{20}{v}\text{ h} \quad \left[\because \text{Time} = \frac{\text{Distance}}{\text{Speed}}\right]$$

Time taken for next 60 km with speed 2v km/h

$$= \frac{60}{2v} = \frac{30}{v}\text{h}$$

$$\therefore \text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

$$= \frac{60 + 60}{\frac{20}{v} + \frac{30}{v}}$$

$$= \frac{120\,v}{50}$$

$= 2.4\ v$ km/h

120. The mean weight of 150 students in a certain class is 60 kg. The mean weight of boys is 70 kg and that of girls is 55 kg. What are the number of boys and girls respectively in the class?

(a) 75 and 75

(b) 50 and 100

(c) 70 and 80

(d) 100 and 50

(*b*) Let number of boys and girls be x and y respectively.

$\therefore \quad x + y = 150 \quad \dots$ (i)

Mean weight of 150 students = 60 kg

$\therefore$ Total weight of 150 students

$= 60 \times 150$

$= 9000$kg.

Mean weight of boys = 70 kg

$\therefore$ Total weight of boys = $70x$ kg

and mean weight of girls = 55 kg

$\therefore$ Total weight of girls = $55\,y$ kg

$\therefore$ Total weight of 150 students = 9000 kg

$\Rightarrow \quad 70x + 55y = 9000$

$\Rightarrow \quad 14x + 11y = 1800 \quad \dots$ (ii)

Solving Eqs. (i) and (ii), we get

$x = 50$

$y = 100$

Hence, the number of boys and girls are 50 and 100 respectively.