NDA/NA

National Defence Academy/Naval Academy

SOLVED PAPER 2018 (I)

PAPER I : Mathematics

1. If $n \in N$, then $121^n - 25^n + 190^n - (-4)^n$ is divisible by which one of the following?

(a) 1904 (b) 2000 (c) 2002 (d) 2006

(b) We have,

$$121^n - 25^n + 1900^n - (-4)^n$$

On putting $n = 1$, we get

$$(121)^1 - (25)^1 + (1900)^1 - (-4)^1$$
$$= 121 - 25 + 1900 + 4$$
$$= 2000$$

Which is divisible by 2000.

2. If $n = (2017)!$, then what is $\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \dots + \frac{1}{\log_{2017} n}$ equal to ?

(a) 0 (b) 1 (c) $\frac{n}{2}$ (d) n

(b) We have,

$$\frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \frac{1}{\log_4 n} + \dots + \frac{1}{\log_{2017} n}$$
$$= \log_n 2 + \log_n 3 + \log_n 4 + \dots + \log_n 2017$$
$$\left[\because \log_b a = \frac{1}{\log_a b}\right]$$
$$= \log_n(2 \cdot 3 \cdot 4 \dots\dots 2017)$$
$$[\because \log a + \log b = \log ab]$$
$$= \log_n(1 \cdot 2 \cdot 3 \cdot 4 \dots\dots 2017)$$
$$= \log_n(2017)!$$
$$[\because n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1 = n!]$$
$$= \log_{(2017)!}(2017)! \quad [\because n = 2017!]$$
$$= 1 \quad [\because \log_a a = 1]$$

3. In the expansion of $(1 + x)^{43}$, if the coefficients of $(2r + 1)$th and $(r + 2)$ th terms are equal, then what is the values of $r (r \neq 1)$?

(a) 5 (b) 14 (c) 21 (d) 22

(b) We have, $(1 + x)^{43}$

$\therefore$ General term, $T_{r+1} = {}^{43}C_r x^r$

Now, $T_{2r+1} = {}^{43}C_{2r} x^{2r}$

and $T_{r+2} = {}^{43}C_{r+1} x^{r+1}$

Now, according to the question

Coefficients of $(2r + 1)$th and $(r + 2)$th terms are equal

$$\therefore \quad {}^{43}C_{2r} = {}^{43}C_{r+1}$$
$$\Rightarrow \quad 2r + r + 1 = 43$$
$$[\because \text{if } {}^nC_x = {}^nC_y \Rightarrow x + y = n]$$
$$\Rightarrow \quad 3r = 42$$
$$\Rightarrow \quad r = 14$$

4. What is the principal argument of $(-1 - i)$, where $i = \sqrt{-1}$?

(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$

(c) $-\frac{3\pi}{4}$ (d) $\frac{3\pi}{4}$

(c) Let $z - 1 - i$

Now, $\tan\alpha = \frac{|b|}{|a|} = \frac{|-1|}{|-1|}$

$$[\because a = -1, b = -1]$$
$$\therefore \quad \alpha = \tan^{-1}(1) = \frac{\pi}{4}$$

Since a, b both are negative,

$$\therefore \arg(z) = \alpha - \pi$$
$$= \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$$

5. Let α and β be real number and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct non-real roots with $\text{Re}(z) = 1$, then it is necessary that

(a) $\beta \in (-1, 0)$ (b) $|\beta| = 1$

(c) $\beta \in (1, \infty)$ (d) $\beta \in (0, 1)$

(c) Let $z = x + iy$

Now, we have

$$z^2 + \alpha z + \beta = 0$$
$$\Rightarrow (x + iy)^2 + \alpha(x + iy) + \beta = 0$$
$$\Rightarrow x^2 - y^2 + 2ixy + \alpha x + i\alpha y + \beta = 0$$
$$[\because i^2 = -1]$$
$$\Rightarrow (x^2 - y^2 + \alpha x + \beta) + (2xy + \alpha y)\, i = 0$$

On comparing,

$$x^2 - y^2 + \alpha x + \beta = 0 \text{ and } 2xy + \alpha y = 0$$
$$\Rightarrow x^2 - y^2 + \alpha x + \beta = 0 \text{ and } (2x + \alpha)y = 0$$
$$\Rightarrow x^2 - y^2 + \alpha x + \beta = 0 \text{ and } 2x + \alpha = 0$$
$$[\because y \neq 0]$$
$$\Rightarrow x^2 - y^2 + \alpha x + \beta = 0 \text{ and } \alpha = -2x$$
$$\Rightarrow x^2 - y^2 + \alpha x + \beta = 0 \text{ and } \alpha = -2$$
$$[\because \text{Re}(z) = 1 = x]$$
$$\Rightarrow 1 - y^2 - 2 + \beta = 0 \quad [\because x = 1, \alpha = -2]$$
$$\Rightarrow \quad \beta = y^2 + 1$$
$$\Rightarrow \quad \beta \in (1, \infty)$$
$$[\because y^2 \geq 0 \Rightarrow y^2 + 1 \geq 1]$$

6. Let A and B be subsets of X and $C = (A \cap B') \cup (A' \cap B)$, where A' and B' are complements of A and B respectively in X. What is C equal to ?

(a) $(A \cup B') - (A \cap B')$

(b) $(A' \cup B) - (A' \cap B)$

(c) $(A \cup B) - (A \cap B)$

(d) $(A' \cup B') - (A' \cap B')$

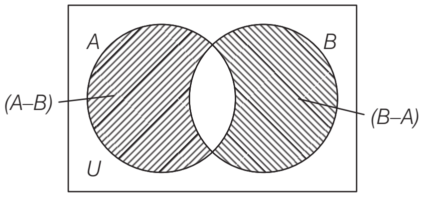
(c) We have, $C = (A \cap B') \cup (A' \cap B)$

$$= (A - B) \cup (B - A)$$

$[\because X \cap Y' = x - y]$

$$= (A \cup B) - (A \cap B)$$

[from venn diagram]



7. How many numbers between 100 and 1000 can be formed with the digits 5, 6, 7, 8, 9, if the repetition of digits is not allowed?

(a) 3^5 (b) 5^3
(c) 120 (d) 60

(d) Number lying between 100 and 1000 are of three digit. Since the numbers are to be formed with 5, 6, 7, 8, 9 and repetition is not allowed, so total number of numbers

$$= 5 \times 4 \times 3 = 60$$

8. The number of non-zero integral solution of the equation $|1 - 2i|^x = 5^x$ is

(a) Zero (no solution)
(b) One
(c) Two
(d) Three

(a) We have,

$$|1 - 2i|^x = 5^x$$

$$\Rightarrow \left(\sqrt{(1)^2 + (-2)^2}\right)^x = 5^x$$

$[\because |a + ib| = \sqrt{a^2 + b^2}]$

$$\Rightarrow (\sqrt{1 + 4})^x = 5^x$$

$$\Rightarrow (\sqrt{5})^x = 5^x$$

$$\Rightarrow 5^{x/2} = 5^x$$

$$\Rightarrow \frac{x}{2} = x$$

$[\because a^m = a^n \Rightarrow m = n]$

$$\Rightarrow x - \frac{x}{2} = 0$$

$$\Rightarrow \frac{x}{2} = 0$$

$$\Rightarrow x = 0$$

But x is non-zero integral.
∴ Given equation has no solution.

9. If the ratio of AM of GM of two positive numbers a and b is 5 : 3, then $a : b$ is equal to

(a) 3 : 5 (b) 2 : 9
(c) 9 : 1 (d) 5 : 3

(c) Let a and b be two numbers.

According to the question,

$$\frac{\frac{a+b}{2}}{\sqrt{ab}} = \frac{5}{3}$$

$$\left[\because A : G = 5 : 3,\ A = \frac{a+b}{2},\ G = \sqrt{ab}\right]$$

$$\Rightarrow \frac{a+b}{\sqrt{ab}} = \frac{10}{3}$$

$$\Rightarrow \frac{(a+b)^2}{ab} = \left(\frac{10}{3}\right)^2$$

$$\Rightarrow \frac{a^2 + b^2 + 2ab}{ab} = \frac{100}{9}$$

$$\Rightarrow \frac{a}{b} + \frac{b}{a} + 2 = \frac{100}{9}$$

$$\Rightarrow t + \frac{1}{t} + 2 = \frac{100}{9} \quad \left[\because \frac{a}{b} = t\right]$$

$$\Rightarrow \frac{t^2 + 1 + 2t}{t} = \frac{100}{9}$$

$$\Rightarrow 9t^2 - 82t + 9 = 0$$

$$\Rightarrow (t - 9)(9t - 1) = 0$$

$$\Rightarrow t = 9, \frac{1}{9}$$

$$\therefore \frac{a}{b} = 9 \text{ or } \frac{a}{b} = \frac{1}{9} \quad \left[\because t = \frac{a}{b}\right]$$

$$\Rightarrow a : b = 9 : 1 \text{ or } 1 : 9$$

10. If the coefficients of a^m and a^n is the expansion of $(1 + a)^{m+n}$ are α and β, then which one of the following is correct?

(a) $\alpha = 2\beta$ (b) $\alpha = \beta$
(c) $2\alpha = \beta$ (d) $\alpha = (m + n)\beta$

(b) We have

$$(1 + a)^{m+n}$$

$$\therefore T_{r+1} = {}^{m+n}C_r a^r$$

∴ Coefficient of $a^m = {}^{m+n}C_m$ $[\because r = m]$

and coefficient of $a^n = {}^{m+n}C_n$ $[\because r = n]$

$$\therefore \alpha = {}^{m+n}C_m$$

and $\beta = {}^{m+n}C_n$

$$= {}^{m+n}C_{m+n-n} \quad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= {}^{m+n}C_m = \alpha$$

$$\therefore \alpha = \beta$$

11. If $x + \log_{15}(1 + 3^x) = x\log_{15} 5 + \log_{15} 12$, where x is an integer, then what is x equal to?

(a) – 3 (b) 2 (c) 1 (d) 3

(c) We have,

$$x + \log_{15}(1 + 3^x) = x\log_{15} 5 + \log_{15} 12$$

$$\Rightarrow \log_{15} 15^x + \log_{15}(1 + 3^x) = \log_{15} 5^x + \log_{15} 12$$

$[\because \log_a a = 1 \text{ and } \log_b a^m = m\log_b a]$

$$\Rightarrow \log_{15}[15^x (1 + 3^x)] = \log_{15}(5^x \times 12)$$

$[\because \log a + \log b = \log ab]$

$$\Rightarrow 15^x (1 + 3^x) = 12 \cdot 5^x$$

$$\Rightarrow 3^x(1 + 3^x) = 12$$

$$\Rightarrow y(1 + y) = 12 \quad [\text{where } y = 3^x]$$

$$\Rightarrow y^2 + y - 12 = 0$$

$$\Rightarrow (y + 4)(y - 3) = 0$$

$$\Rightarrow y = -4, 3$$

$$\Rightarrow 3^x = -4, 3$$

$$\Rightarrow 3^x = 3 \quad [\because 3^x \neq -4]$$

$$\Rightarrow x = 1$$

12. How many four-digit numbers divisible by 10 can be formed using 1, 5, 0, 6, 7 without repetition of digits?

(a) 24 (b) 36
(c) 44 (d) 64

(a) We have to form four digit numbers which are divisible by 10 and using 1, 5, 0, 6, 7. Since numbers must be divisible by 10, so unit place must be zero.

∴ Total number of such numbers = Permutations of three digits using 1, 5, 6, 7

$$= {}^4P_3 = \frac{4!}{(4-3)!} = 4! = 24$$

Directions (Q. Nos. 13-14) *Consider the information given below and answer the two items that follow*

In a class, 54 students are good in Hindi only, 63 students are good in Mathematics only and 41 students are good in English only. There are 18 students who are good in both Hindi and Mathematics. 10 students are good in all three subjects.

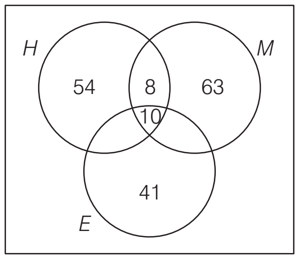
13. What is the number of students who are good in either Hindi or Mathematics but not English?

(a) 99 (b) 107
(c) 125 (d) 130

14. What is the number of students who are good in Hindi and Mathematics but not in English?

(a) 18 (b) 12 (c) 10 (d) 8

Solution (Q. Nos. 13-14)



(c) From given Venn-diagram
$n(H \cup M \cup E') = 54 + 8 + 63 = 125$

(d) From given Venn-diagram
$n(H \cap M \cap E') = 18 - 10 = 8$

15. If α and β are different complex numbers with $|\alpha| = 1$, then what is $\left|\frac{\alpha - \beta}{1 - \alpha\bar{\beta}}\right|$ equal to ?

(a) $|\beta|$ (b) 2 (c) 1 (d) 0

(c) We have,

$$\left|\frac{\alpha - \beta}{1 - \alpha\bar{\beta}}\right| = \left|\frac{\alpha - \beta}{\alpha\bar{\alpha} - \alpha\bar{\beta}}\right|$$

$$[\because |\alpha| = 1 \Rightarrow |\alpha|^2 = 1 \Rightarrow \alpha \cdot \bar{\alpha} = 1]$$

$$= \left|\frac{\alpha - \beta}{\alpha(\bar{\alpha} - \bar{\beta})}\right|$$

$$= \frac{1}{|\alpha|}\left|\frac{\alpha - \beta}{\bar{\alpha} - \bar{\beta}}\right|$$

$$= \frac{|\alpha - \beta|}{|\alpha||\overline{\alpha - \beta}|}$$

$$= \frac{|\alpha - \beta|}{|\alpha||\alpha - \beta|} \quad [\because |\bar{z}| = |z|]$$

$$= \frac{1}{|\alpha|} = 1$$

$[\because |\alpha| = 1]$

16. The equation $|1 - x| + x^2 = 5$ has

(a) a rational root and an irrational root
(b) two rational roots
(c) two irrational roots
(d) no real roots

(a) We have,

$$|1 - x| + x^2 = 5$$

Case I When $x < 1$

$$1 - x + x^2 = 5$$

$$[\because x < 1 \Rightarrow |1 - x| = 1 - x]$$

$$\Rightarrow x^2 - x - 4 = 0$$

$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1 - 4(1)(-4)}}{2}$$

$$= \frac{1 \pm \sqrt{17}}{2}$$

$$\Rightarrow \quad x = \frac{1 - \sqrt{17}}{2} \quad [\because x < 1]$$

Case II When $x \geq 1$

$$-(1 - x) + x^2 = 5$$

$$[\because x \geq 1 \Rightarrow |1 - x| = -(1 - x)]$$

$$\Rightarrow -1 + x + x^2 = 5$$

$$\Rightarrow \quad x^2 + x - 6 = 0$$

$$\Rightarrow (x + 3)(x - 2) = 0$$

$$\Rightarrow \quad x = -3, 2$$

$$\Rightarrow \quad x = 2 \quad [\because x \geq 1]$$

$\therefore$ Given equation has a rational root and an irrational root.

17. The binary number expression of the decimal number 31 is

(a) 1111 (b) 10111
(c) 11011 (d) 11111

(d)

2	31	
2	15	1
2	7	1
2	3	1
2	1	1
	0	1

$\therefore (31)_{10} = (11111)_2$

18. What is $i^{1000} + i^{1001} + i^{1002} + i^{1003}$ equal to (where $i = \sqrt{-1}$) ?

(a) 0 (b) i (c) $-i$ (d) 1

(a) We have,

$$i^{1000} + i^{1001} + i^{1002} + i^{1003}$$

$$= i^{1000}[1 + i + i^2 + i^3]$$

$$= i^{1000}[1 + i - 1 - i]$$

$$[\because i^2 = -1, i^3 = -i]$$

$$= 0$$

19. What is $\frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \frac{1}{\log_4 N} + \ldots + \frac{1}{\log_{100} N}$ equal to $(N \neq 1)$?

(a) $\frac{1}{\log_{100!} N}$ (b) $\frac{1}{\log_{99!} N}$

(c) $\frac{99}{\log_{100!} N}$ (d) $\frac{99}{\log_{99!} N}$

(a) We have,

$$\frac{1}{\log_2 N} + \frac{1}{\log_3 N} + \frac{1}{\log_4 N} + \ldots + \frac{1}{\log_{100} N}$$

$$= \log_N 2 + \log_N 3 + \log_N 4 + \ldots + \log_N 100$$

$$\left[\because \log_b a = \frac{1}{\log_a b}\right]$$

$$= \log_N 2 \cdot 3 \cdot 4 \ldots 100$$

$$[\because \log a + \log b = \log ab]$$

$$= \log_N 1 \cdot 2 \cdot 3 \cdot 4 \ldots 100$$

$$= \log_N (100)!$$

$$[\because n! = n(n-1)(n-2)\ldots 2 \cdot 1]$$

$$= \frac{1}{\log_{(100)!} N}$$

20. The modulus-amplitude form of $\sqrt{3} + i$, where $i = \sqrt{-1}$ is

(a) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(b) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(c) $4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(d) $4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(b) Let $z = \sqrt{3} + i$

$$\therefore \quad |z| = \sqrt{(\sqrt{3})^2 + (1)^2}$$

$$[\because z = a + ib \Rightarrow |z| = \sqrt{a^2 + b^2}]$$

$$= \sqrt{3 + 1} = \sqrt{4} = 2$$

Now, $\text{amp}(z) = \tan^{-1}\left(\frac{b}{a}\right)$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\left[\because \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}\right]$$

$$\therefore \quad z = r(\cos\theta + i\sin\theta)$$

$$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$[\because r = |z| = 2 \text{ and } \theta = \text{amp}(z) = \frac{\pi}{6}]$$

21. What is the number of non-zero terms in the expansion of $(1 + 2\sqrt{3}x)^{11} + (1 - 2\sqrt{3}x)^{11}$ (after simplification)?

(a) 4 (b) 5
(c) 6 (d) 11

(c) In $(a + b)^n + (a - b)^n$, number of terms

$$= \begin{cases} \frac{n+2}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ Number of terms in $(1 + 2\sqrt{3}x)^{11} + (1 - 2\sqrt{3}x)^{11}$

$$= \frac{11 + 1}{2} \quad [\because n = 11, \text{ is odd}]$$

$$= \frac{12}{2} = 6$$

22. What is the greatest integer among the following, by which the number $5^5 + 7^5$ is divisible?

(a) 6 (b) 8 (c) 11 (d) 12

(d) We know that when m is odd then $(x^m + y^m)$ is divisible by $(x + y)$.

$\therefore 5^5 + 7^5$ is divisible by $5 + 7 = 12$ as $m = 5$ is odd.

23. If $x = 1 - y + y^2 - y^3 \ldots$ up to infinite terms, where $|y| < 1$, then which one of the following is correct?

(a) $x = \frac{1}{1 + y}$ (b) $x = \frac{1}{1 - y}$

(c) $x = \frac{y}{1 + y}$ (d) $x = \frac{y}{1 - y}$

(a) We have,

$$x = 1 - y + y^2 - y^3 + \ldots \infty, |y| < 1$$

$$= \frac{1}{1 - (-y)}$$

$$\left[\because a + ar + ar^2 + \ldots \infty = \frac{a}{1 - r}, r < 1\right]$$

$$= \frac{1}{1 + y}$$

24. What is the inverse of the matrix $A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$?

(a) $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

(d) $\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(a) We have,

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|A| = 1[\cos^2\theta - (-\sin^2\theta)] = 1 \neq 0$

$$C_{11} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta$$

$$C_{12} = -\begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta$$

$$C_{13} = \begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{21} = -\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} = -\sin\theta$$

$$C_{22} = \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta$$

$$C_{23} = -\begin{vmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} \sin\theta & 0 \\ \cos\theta & 0 \end{vmatrix} = 0$$

$$C_{32} = -\begin{vmatrix} \cos\theta & 0 \\ -\sin\theta & 0 \end{vmatrix} = 0$$

$$C_{33} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$\therefore \text{adj}A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{adj}$$

$$A = \frac{1}{1} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

25. If A is a 2×3 matrix and AB is a 2×5 matrix, then B must be a

(a) 3×5 matrix (b) 5×3 matrix
(c) 3×2 matrix (d) 5×2 matrix

(a) Let order of B is $m \times n$.

Now, according to the question

$A_{2 \times 3} \times B_{m \times n} = (AB)_{2 \times 5}$

$\therefore$ $m = 3$ and $n = 5$

$\therefore$ Order of B is 3×5.

26. If $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $A^2 - kA - I_2 = O$, where I_2 is the 2×2 identity matrix, then what is the value of k ?

(a) 4 (b) – 4
(c) 8 (d) – 8

(a) We have,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\therefore \quad A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot1 + 2\cdot2 & 1\cdot2 + 2\cdot3 \\ 2\cdot1 + 3\cdot2 & 2\cdot2 + 3\cdot3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$$

Now, it is given that,

$A^2 - kA - I_2 = 0$

$$\Rightarrow \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} k & 2k \\ 2k & 3k \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 4 & 8 \\ 8 & 12 \end{bmatrix} = \begin{bmatrix} k & 2k \\ 2k & 3k \end{bmatrix}$$

$\Rightarrow \quad k = 4$

27. What is the number of triangles that can be formed by choosing the vertices from a set of 12 points in a plane, seven of which lie on the same straight line?

(a) 185 (b) 175
(c) 115 (d) 105

(a) Required number of triangle

$= {}^{12}C_3 - {}^7C_3$

$$= \frac{12 \times 11 \times 10}{3 \times 2 \times 1} - \frac{7 \times 6 \times 5}{3 \times 2 \times 1}$$

$= 220 - 35 = 185$

28. What is $C(n, r) + 2C(n, r-1) + C(n, r-2)$ equal to ?

(a) $C(n + 1, r)$
(b) $C(n - 1, r + 1)$
(c) $C(n, r + 1)$
(d) $C(n + 2, r)$

(d) We have,

$C(n, r) + 2C(n, r-1) + C(n, r-2)$

$= {}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2}$

$= ({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})$

$= {}^{n+1}C_r + {}^{n+1}C_{r-1}$

$[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$= {}^{n+1+1}C_r$

$= {}^{n+2}C_r$

$= C(n + 2, r)$

29. Let $|x|$ denote the greatest integer function. What is the number of solutions of the equation $x^2 - 4x + [x] = 0$ in the interval $[0, 2]$?

(a) Zero (no solution) (b) One
(c) Two (d) Three

(b) We have,

$x^2 - 4x + [x] = 0$

Case I $x \in [0, 1]$

$\therefore \quad x^2 - 4x + 0 = 0$

$[\because x \in [0, 1] \Rightarrow [x] = 0]$

$\Rightarrow \quad x^2 - 4x = 0$

$\Rightarrow \quad x(x - 4) = 0$

$\Rightarrow \quad x = 0, 4$

$\Rightarrow \quad x = 0$

$[\because x \in [0, 1]]$

Case II $x \in [1, 2]$

$\therefore \quad x^2 - 4x + 1 = 0$

$[\because x \in [1, 2] \Rightarrow x = 1]$

$$\Rightarrow \quad x = \frac{4 \pm \sqrt{16 - 4}}{2}$$

$$\Rightarrow \quad x = \frac{4 \pm 2\sqrt{3}}{2}$$

$\Rightarrow \quad x = 2 \pm \sqrt{3}$

$\Rightarrow \quad x = 0.268, 3.732$

No solution $[\because x \in [1,2]]$

$\therefore$ Given equation has only one solution i.e. $x = 0$.

30. A survey of 850 students in a University yields that 680 students like music and 215 like dance. What is the least number of students who like both music and dance?

(a) 40 (b) 45
(c) 50 (d) 55

(b) Let A be the set of students who like music and B be the set of students whose like dance.

$\therefore n(A) = 680$, $n(B) = 215$ and $n(U) = 850$

We know that,

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$

$\Rightarrow$

$$n(A \cap B)_{\min} = n(A) + n(B) - n(A \cup B)_{\max}$$

$\Rightarrow n(A \cap B)_{min} = 680 + 215 - 850$

$[\because n(A \cup B)_{max} = n(\cup)]$

$= 45$

31. What is the sum of all two-digit numbers, which when divided by 3 leave 2 as the remainder ?

(a) 1565 (b) 1585
(c) 1635 (d) 1655

(c) Required numbers are 11, 14, 17, ... 98 which is an AP.

We know that,

$a_n = a + (n-1)d$

$98 = 11 + (n-1)(3)$

$\Rightarrow 98 = 11 + 3n - 3$

$\Rightarrow 98 = 3n + 8$

$\Rightarrow 90 = 3n$

$\Rightarrow n = 30$

$\therefore$ Sum $= 11 + 14 + 17 + \dots + 98$

$= \frac{30}{2}[11 + 98] \quad \left[\because S_n = \frac{n}{2}(a + l)\right]$

$= 15 \times 109$

$= 1635$

32. If $0 > a < 1$, the value of $\log_{10} a$ is negative. This is justified by

(a) Negative power of 10 is less than 1
(b) Negative power of 10 is between 0 and 1
(c) Negative power of 10 is positive
(d) Negative power of 10 is negative

(b) Let $\log_{10} a = x$

$\Rightarrow a = 10^x$

It is given that

$0 < a < 1$

$\Rightarrow 0 < 10^x < 1$

$\Rightarrow x$ must be negative

$\therefore$ If $0 < a < 1$, the value of $\log_{10} a$ is negative implies that negative power of 10 is between 0 and 1.

33. The third term of a GP is 3. What is the product of the first five terms?

(a) 216
(b) 226
(c) 243
(d) Cannot be determined due to insufficient data

(c) Let a and r be the first term and common ratio of of the GP.

$\therefore a_3 = 3$

$\Rightarrow ar^2 = 3 \quad [\because a_n = ar^{n-1}] \quad \dots$ (i)

Required product $= a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5$

$= (a)(ar)(ar^2)(ar^3)(ar^4)$

$= a^5 r^{10} = (ar^2)^5$

$= (3)^5$ [from Eq. (i)]

$= 243$

34. If $x, \frac{3}{2}, z$ are in AP; $x, 3, z$ are in GP; then which one of the following will be in HP?

(a) x, 6, z (b) x, 4, z
(c) x, 2, z (d) x, 1, z

(a) We have,

$x, \frac{3}{2}, z$ are in AP.

$\Rightarrow \frac{x + z}{2} = \frac{3}{2}$

$\Rightarrow x + z = 3 \quad \dots$ (i)

Also, x, 3, z are in GP

$\Rightarrow xz = 3^2$

$\Rightarrow xz = 9 \quad \dots$ (ii)

Now, from Eqs. (i) and (ii), we have

$\frac{2xz}{x + y} = \frac{2 \times 9}{3}$

$\Rightarrow \frac{2xz}{x + z} = 6$

$\Rightarrow x$, 6, z are in HP.

35. What is the value of the sum $\sum_{n=2}^{11}(i^n + i^{n+1})$, where $i = \sqrt{-1}$?

(a) i (b) $2i$ (c) $-2i$ (d) $1 + i$

(c) We have,

$\sum_{n=2}^{11}(i^n + i^{n+1}) = \sum_{n=2}^{11} i^n(1 + i)$

$= (1 + i)\sum_{n=2}^{11} i^n$

$= (1 + i)[i^2 + i^3 + i^4 + \dots + i^{11}]$

$= (1 + i)\, i^2\left[\frac{i^{10} - 1}{i - 1}\right]$

$\left[\because a + ar + ar^2 + \dots + ar^{n-1} = a\left[\frac{r^n - 1}{r - 1}\right]\right]$

$= \frac{(1 + i)i^2(i^{2 \times 4 + 2} - 1)}{(i - 1)}$

$= \frac{-(1 + i)(i^2 - 1)}{(i - 1)} \quad [\because i^2 = -1]$

$= \frac{-(1 + i)(-1 - 1)}{(i - 1)} = \frac{2(1 + i)}{(i - 1)}$

$= \frac{2(1 + i)}{(i - 1)} \times \frac{i + 1}{(i + 1)}$

$= \frac{2(i + 1 + i^2 + i)}{i^2 - 1}$

$= \frac{2(i + 1 - 1 + i)}{-1 - 1}$

$= -2i$

36. If $\sin x = \frac{1}{\sqrt{5}}$, $\sin y = \frac{1}{\sqrt{10}}$, where $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$, then what is $(x + y)$ equal to ?

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) 0

(c) We have, $\sin x = \frac{1}{\sqrt{5}}$ and $\sin y = \frac{1}{\sqrt{10}}$

$\Rightarrow x = \sin^{-1}\frac{1}{\sqrt{5}}$ and $y = \sin^{-1}\frac{1}{\sqrt{10}}$

Now, $x + y = \sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}}$

$= \sin^{-1}\left[\frac{1}{\sqrt{5}}\sqrt{1 - \left(\frac{1}{\sqrt{10}}\right)^2} + \frac{1}{\sqrt{10}}\sqrt{1 - \left(\frac{1}{\sqrt{5}}\right)^2}\right]$

$[\because \sin^{-1} x + \sin^{-1} y = \sin^{-1}[x\sqrt{1 - y^2} + y\sqrt{1 - x^2}]$

$= \sin^{-1}\left[\frac{1}{\sqrt{5}}\sqrt{1 - \frac{1}{10}} + \frac{1}{\sqrt{10}}\sqrt{1 - \frac{1}{5}}\right]$

$= \sin^{-1}\left[\frac{1}{\sqrt{5}} \times \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}} \times \frac{2}{\sqrt{5}}\right]$

$= \sin^{-1}\left[\frac{5}{\sqrt{5} \times \sqrt{10}}\right]$

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4} \quad \left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$

37. What is $\frac{\sin 5x - \sin 3x}{\cos 5x + \cos 3x}$ equal to ?

(a) $\sin x$ (b) $\cos x$
(c) $\tan x$ (d) $\cot x$

(c) Given, $\frac{\sin 5x - \sin 3x}{\cos 5x + \cos 3x}$

$= \frac{2\cos\frac{5x + 3x}{2} \cdot \sin\frac{5x - 3x}{2}}{2\cos\frac{5x + 3x}{2} \cdot \cos\frac{5x - 3x}{2}}$

$\left[\because \sin C - \sin D = 2\cos\left(\frac{C + D}{2}\right) \cdot \sin\left(\frac{C - D}{2}\right) \text{ and } \cos C + \cos D = 2\cos\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)\right]$

$= \frac{2\cos 4x \sin x}{2\cos 4x \cos x} = \frac{\sin x}{\cos x} = \tan x$

38. What is $\sin 105° + \cos 105°$ equal to ?

(a) $\sin 50°$ (b) $\cos 50°$
(c) $\frac{1}{\sqrt{2}}$ (d) 0

(c) We have, $\sin 105° + \cos 105°$

$= \sin(90° + 15°) + \cos 105°$

$= \cos 15° + \cos 105°$

$[\because \sin(90 + \theta) = \cos\theta]$

$= 2\cos\left(\frac{105° + 15°}{2}\right)\cos\left(\frac{105° - 15°}{2}\right)$

$\left[\because \cos C + \cos D = 2\cos\left(\frac{C + D}{2}\right)\cos\left(\frac{C - D}{2}\right)\right]$

$= 2\cos 60° \cos 45° = 2 \times \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$$\left[\because \cos 60° = \frac{1}{2}, \cos 45° = \frac{1}{\sqrt{2}}\right]$$

$= \frac{1}{\sqrt{2}}$

39. In a ΔABC, if $a = 2$, $b = 3$ and $\sin A = \frac{2}{3}$, then what is $\angle B$ equal to ?

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

(b) We have, $a = 2$, $b = 3$ and $\sin A = \frac{2}{3}$

Now, from sine formula

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$\Rightarrow \quad \frac{\frac{2}{3}}{2} = \frac{\sin B}{b}$

$\Rightarrow \quad \sin B = 1$

$\Rightarrow \quad B = \frac{\pi}{2} \quad \left[\because \sin\frac{\pi}{2} = 1\right]$

40. What is the principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$?

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

(c) We have, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

$= \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right)$

$= \sin^{-1}\sin\frac{\pi}{3} \quad [\because \sin(\pi - \theta) = \sin\theta]$

$= \frac{\pi}{3}$

$$\left[\because \sin^{-1}\sin\theta = \theta, \text{ if } \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right]$$

41. If x, $x - y$ and $x + y$ are the angles of a triangle (not an equilateral triangle) such that $\tan(x - y)$, $\tan x$ and $\tan(x + y)$ are in GP, then what is x equal to ?

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

(b) We have,

$x, x - y, x + y$ are the angles of a triangle. Since, sum of angles of a triangle $= \pi$

$\therefore \quad x + x - y + x + y = \pi$

$\Rightarrow \quad 3x = \pi$

$\Rightarrow \quad x = \frac{\pi}{3}$

42. ABC is a triangle inscribed in a circle with centre O. Let $\alpha = \angle BAC$, where $45° < \alpha < 90°$. Let $\beta = \angle BOC$. Which one of the following is correct?

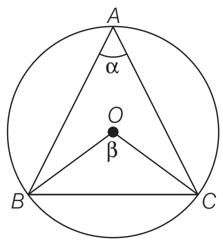
(a) $\cos\beta = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$

(b) $\cos\beta = \frac{1 + \tan^2\alpha}{1 - \tan^2\alpha}$

(c) $\cos\beta = \frac{2\tan\alpha}{1 + \tan^2\alpha}$

(d) $\sin\beta = 2\sin^2\alpha$

(a) We know that angle subtended by a chord at centre is always double the angle subtended by it at any other part of the circle.



$\therefore \quad \beta = 2\alpha$

$\Rightarrow \quad \cos\beta = \cos 2\alpha$

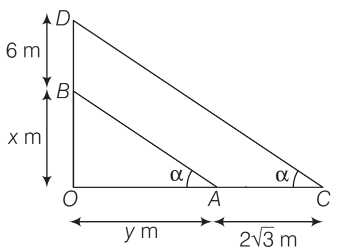
$\Rightarrow \quad \cos\beta = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$

$$\left[\because \cos 2\theta = \frac{1 + \tan^2\theta}{1 + \tan^2\theta}\right]$$

43. If a flage-staff of 6 m height placed on the top of a tower throws a shadow of $2\sqrt{3}$ m along the ground, then what is the angle that the sun makes with the ground ?

(a) 60° (b) 45°
(c) 30° (d) 15°

(a) Let OB and BD be the tower and flag-staff respectively. OA and AC be the shadow of tower and flag-staff respectively.



Again let α be the angle that sun makes with the ground.

$\therefore \quad \angle OAB = \angle OCD = \alpha$

Now, in ΔOAB

$$\tan\alpha = \frac{x}{y} \quad \ldots \text{(i)}$$

and in ΔOCD

$$\tan\alpha = \frac{x + 6}{y + 2\sqrt{3}} \quad \ldots \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{x}{y} = \frac{x + 6}{y + 2\sqrt{3}}$$

$\Rightarrow \quad xy + 2\sqrt{3}x = xy + 6x$

$\Rightarrow \quad \frac{x}{y} = \sqrt{3}$

$\Rightarrow \quad \tan\alpha = \sqrt{3} \quad$ [from Eq. (i)]

$\Rightarrow \quad \alpha = 60°$

44. What is $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right)$ equal to ?

(a) 0 (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

(b) We have

$\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right)$

$= \tan^{-1}\left[\frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4} \times \frac{3}{5}}\right]$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x + y}{1 - xy}, xy < 1\right]$$

$= \tan^{-1}\left[\frac{\frac{5 + 12}{20}}{\frac{20 - 3}{20}}\right]$

$= \tan^{-1}\left(\frac{17}{17}\right) = \tan^{-1} 1$

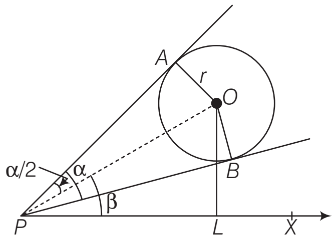
$= \frac{\pi}{4} \quad \left[\because \tan^{-1} 1 = \frac{\pi}{4}\right]$

45. A spherical balloon of radius r subtends an angle α at the eye of an observer, while the angle of elevation of its centre is β. What is the height of the centre of the balloon (neglecting the height of the observer)?

(a) $\frac{r\sin\beta}{\sin\left(\frac{\alpha}{2}\right)}$ (b) $\frac{r\sin\beta}{\sin\left(\frac{\alpha}{4}\right)}$

(c) $\frac{r\sin\left(\frac{\beta}{2}\right)}{\sin\alpha}$ (d) $\frac{r\sin\alpha}{\sin\left(\frac{\beta}{2}\right)}$

(a) Let O bet the centre of the balloon, P be the eye of the observer and $\angle APB$ be the angle subtendd by the balloon at the eye of the observer. $\angle APB = \alpha$



$\therefore \quad \angle APO = \angle BPO = \dfrac{\alpha}{2}$

In ΔOAP

$$\sin\frac{\alpha}{2} = \frac{OA}{OP}$$

$$\Rightarrow \sin\frac{\alpha}{2} = \frac{r}{OP} \Rightarrow OP = r \ \text{cosec}\ \frac{\alpha}{z} \quad \dots \text{(i)}$$

In ΔOPL,

$$\sin\beta = \frac{OL}{OP}$$

$$\Rightarrow \quad OL = OP\sin\beta$$

$$\Rightarrow \quad OL = r \ \text{cosec}\frac{\alpha}{2} \cdot \sin\beta$$

[from Eqs. (i)]

$$\therefore \quad OL = \frac{r\sin\beta}{\sin\left(\frac{\alpha}{2}\right)}$$

46. If $\dfrac{\sin(x+y)}{\sin(x-y)} = \dfrac{a+b}{a-b}$, then what is $\dfrac{\tan x}{\tan y}$ equal to ?

(a) $\dfrac{a}{b}$ (b) $\dfrac{b}{a}$

(c) $\dfrac{a+b}{a-b}$ (d) $\dfrac{a-b}{a+b}$

(a) We have,

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$$

On using componendo and dividendo rule, we get

$$\frac{\sin(x+y)+\sin(x-y)}{\sin(x+y)-\sin(x-y)} = \frac{a+b+a-b}{a+b-a+b}$$

$$\Rightarrow \frac{2\sin\left(\frac{x+y+x-y}{2}\right)\cos\left(\frac{x+y-x+y}{2}\right)}{2\cos\left(\frac{x+y+x-y}{2}\right)\sin\left(\frac{x+y-x+y}{2}\right)} = \frac{2a}{2b}$$

$$\Rightarrow \frac{\sin x\cos y}{\cos x\sin y} = \frac{a}{b}$$

$$\Rightarrow \quad \frac{\tan x}{\tan y} = \frac{a}{b}$$

47. If $\sin\alpha + \sin\beta = 0 = \cos\alpha + \cos\beta$, where $0 < \beta < \alpha < 2\pi$, then which one of the following is correct?

(a) $\alpha = \pi - \beta$ (b) $\alpha = \pi + \beta$

(c) $\alpha = 2\pi - \beta$ (d) $2\alpha = \pi + 2\beta$

(b) We have,

$$\sin\alpha + \sin\beta = 0 = \cos\alpha + \cos\beta$$

$$\therefore (\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2 = 0$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta + \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta = 0$$

$$\Rightarrow (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) + 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 0$$

$$\Rightarrow \quad 1 + 1 + 2\cos(\alpha - \beta) = 0$$

$$\Rightarrow \quad 2\cos(\alpha - \beta) = -2$$

$$\Rightarrow \quad \cos(\alpha - \beta) = -1$$

$$\Rightarrow \quad \alpha - \beta = \pi$$

$$\Rightarrow \quad \alpha = \beta + \pi$$

48. Suppose $\cos A$ is given. If only one value of $\cos\left(\dfrac{A}{2}\right)$ is possible, then A must be

(a) An odd multiple of 90°

(b) A multiple of 90°

(c) An odd multiple of 180°

(d) A multiple of 180°

(c) We know that,

$$\cos A = 2\cos^2\frac{A}{2} - 1$$

Since, $\cos A$ is given and $\cos\dfrac{A}{2}$ has only one solution. So, A must be odd multiple of 180°.

49. If $\cos\alpha + \cos\beta + \cos\gamma = 0$, where $0 < \alpha \le \dfrac{\pi}{2}$, $0 < \beta \le \dfrac{\pi}{2}$, $0 < \gamma \le \dfrac{\pi}{2}$, then what is the value of $\sin\alpha + \sin\beta = \sin\gamma$?

(a) 0 (b) 3

(c) $\dfrac{5\sqrt{2}}{2}$ (d) $\dfrac{3\sqrt{2}}{2}$

(b) We have,

$$\cos\alpha + \cos\beta + \cos\gamma = 0,$$

$$0 < \alpha \le \frac{\pi}{2},\ 0 < \beta \le \frac{\pi}{2},\ 0 < \gamma \le \frac{\pi}{2}$$

$$\cos\alpha + \cos\beta + \cos\gamma = 0$$

$$\Rightarrow \quad \alpha = \beta = \gamma = \frac{\pi}{2}$$

$$\therefore \quad \sin\alpha + \sin\beta + \sin\gamma = \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2}$$

$$\left[\because \text{ if } \theta \in \left[0, \frac{\pi}{2}\right] \cos\frac{\pi}{2} = 0\right]$$

$$= 1 + 1 + 1 = 3$$

50. The maximum value of $\sin\left(x + \dfrac{\pi}{5}\right) + \cos\left(x + \dfrac{\pi}{5}\right)$, where $x \in \left(0, \dfrac{\pi}{2}\right)$, is attained at

(a) $\dfrac{\pi}{20}$ (b) $\dfrac{\pi}{15}$ (c) $\dfrac{\pi}{10}$ (d) $\dfrac{\pi}{2}$

(a) Let $f(x) = \sin\left(x + \dfrac{\pi}{5}\right) + \cos\left(x + \dfrac{\pi}{5}\right)$

$$= \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin\left(x + \frac{\pi}{5}\right) + \frac{1}{\sqrt{2}}\cos\left(x + \frac{\pi}{5}\right)\right]$$

$$= \sqrt{2}\left[\sin\left(x + \frac{\pi}{5}\right)\cos\frac{\pi}{4} + \cos\left(x + \frac{\pi}{5}\right)\sin\frac{\pi}{4}\right]$$

$$= \sqrt{2}\left[\sin\left(x + \frac{\pi}{5} + \frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\sin\left(x + \frac{\pi}{5} + \frac{\pi}{4}\right)$$

$f(x)$ attains maximum value, when

$$x + \frac{\pi}{5} + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{\pi}{20}$$

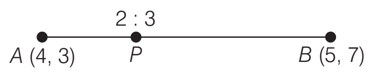
51. What is the distance between the points which divide the line segment joining $(4, 3)$ and $(5, 7)$ internally and externally in the ratio 2 : 3 ?

(a) $\dfrac{12\sqrt{17}}{5}$ (b) $\dfrac{13\sqrt{17}}{5}$

(c) $\dfrac{\sqrt{17}}{5}$ (d) $\dfrac{6\sqrt{17}}{5}$

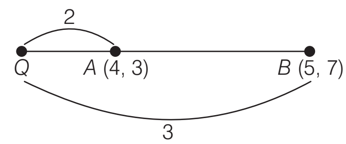
(a) Let P and Q be the points which divides $A(4, 3)$ and $B(5, 7)$ internally and externally in the ratio 2 : 3 respectively.

$$\therefore P = \left(\frac{2 \times 5 + 3 \times 4}{2 + 3}, \frac{2 \times 7 + 3 \times 3}{2 + 3}\right)$$



$$= \left(\frac{22}{5}, \frac{23}{5}\right)$$

and $Q = \left(\dfrac{2 \times 5 - 3 \times 4}{2 + 3}, \dfrac{2 \times 7 - 3 \times 3}{2 - 3}\right)$



$$= (2, -5)$$

$\therefore$ Required distance $= PQ$

$$= \sqrt{\left(2 - \frac{22}{5}\right)^2 + \left(-5 - \frac{23}{5}\right)^2} = \frac{12}{5}\sqrt{17}$$

52. What is the angle between the straight lines $(m^2 - mn)y = (mn + n^2)x + n^3$ and $(mn + m^2)y = (mn - n^2)x + m^3$, where $m > n$?

(a) $\tan^{-1}\left(\dfrac{2mn}{m^2 + n^2}\right)$ (b) $\tan^{-1}\left(\dfrac{4m^2n^2}{m^2 - n^2}\right)$

(c) $\tan^{-1}\left(\dfrac{4m^2n^2}{m^4 + n^4}\right)$ (d) 45°

(c) Given equations of lines are

$$(m^2 - mn)\, y = (mn + n^2)\, x + n^3$$

and $(mn + m^2)y = (mn - n^2)\, x + m^3$

Given equation of lines can be written as

$$y = \frac{mn + n^2}{m^2 - mn} x + \frac{n^3}{m^2 - mn}$$

and $y = \dfrac{mn - n^2}{mn + m^2} x + \dfrac{m^3}{mn + m^2}$

Let m_1 and m_2 be the slopes of given lines.

$$\therefore \quad m_1 = \frac{mn + n^2}{m^2 - mn}$$

and $m_2 = \dfrac{mn - n^2}{mn + m^2}$

If θ is the angle between these lines, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\dfrac{mn + n^2}{m^2 - mn} - \dfrac{mn - n^2}{mn + m^2}}{1 + \dfrac{mn + n^2}{m^2 - mn} \cdot \dfrac{mn - n^2}{mn + n^2}}$$

$$= \frac{(mn+n^2)(mn+m^2) - (mn-n^2)(m^2-mn)}{(m^2-mn)(mn+m^2) + (mn+n^2)(mn-n^2)}$$

$$= \frac{m^2n^2 + m^3n + mn^3 + m^2n^2 - m^3n + m^2n^2 + m^2n^2 - mn^3}{m^3n + m^4 - m^2n^2 - m^3n^2 - m^2n^2 - mn^3 + mn^3 - n^4}$$

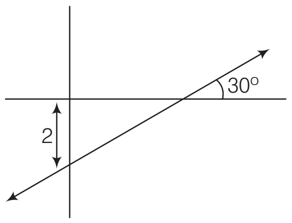
$$= \frac{4m^2n^2}{m^4 - n^4}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{4m^2n^2}{m^4 - n^4}\right)$$

53. What is the equation of the straight line cutting-off an intercept 2 from the negative direction of Y-axis and inclined at 30° with the positive direction of X-axis ?

(a) $x - 2\sqrt{3}y - 3\sqrt{2} = 0$
(b) $x + 2\sqrt{3}y - 3\sqrt{2} = 0$
(c) $x + \sqrt{3}y - 2\sqrt{3} = 0$
(d) $x - \sqrt{3}y - 2\sqrt{3} = 0$

(d) From the given figure, it is clear that



Slope of line $= \tan 30° = \dfrac{1}{\sqrt{3}}$ and line passes through the point $(0, -2)$.

$\therefore$ Equation of line is

$$y - (-2) = \frac{1}{\sqrt{3}}(x - 0)$$

$$\Rightarrow \quad y + 2 = \frac{1}{\sqrt{3}} x$$

$$\Rightarrow \quad \sqrt{3}\, y + 2\sqrt{3} = x$$

$$\Rightarrow \quad x - \sqrt{3}y - 2\sqrt{3} = 0$$

54. What is the equation of the line passing through the point of intersection of the lines $x + 2y - 3 = 0$ and $2x - y + 5 = 0$ and parallel to the line $y - x + 10 = 0$?

(a) $7x - 7y + 18 = 0$
(b) $5x - 7y + 18 = 0$
(c) $5x - 5y + 18 = 0$
(d) $x - y + 5 = 0$

(c) Equation of line passing through intersection point of lines $x + 2y - 3 = 0$ and $2x - y + 5 = 0$ is

$$x + 2y - 3 + \lambda(2x - y + 5) = 0 \quad \text{... (i)}$$

$$\Rightarrow (1 + 2\lambda)x + (2 - \lambda)y + 5\lambda - 3 = 0$$

$\therefore$ Slope of above line $= -\dfrac{(1 + 2\lambda)}{(2 - \lambda)}$

Since line is parallel to $y - x + 10 = 0$

$$= \frac{-(1 + 2\lambda)}{(2 - \lambda)} = \frac{-(-1)}{1}$$

$$\Rightarrow -(1 + 2\lambda) = 2 - \lambda$$

$$\Rightarrow \quad -1 - 2\lambda = 2 - \lambda$$

$$\Rightarrow \quad -\lambda = 3$$

$$\Rightarrow \quad \lambda = -3$$

Putting $\lambda = -3$ in Eq. (i), we get

$$x + 2y - 3 - 3(2x - y + 5) = 0$$

$$\Rightarrow \quad 5x - 5y + 18 = 0$$

Which is equation of required line.

55. Consider the following statements

I. The length p of the perpendicular from the origin to the line $ax + by = c$ satisfies the relation $p^2 = \dfrac{c^2}{a^2 + b^2}$.

II. The length p of the perpendicular from the origin to the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ satisfied the relation $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$.

III. The length p of the perpendicular from the origin to the line $y = mx + c$ satisfies the relation $\dfrac{1}{p^2} = \dfrac{1 + m^2 + c^2}{c^2}$.

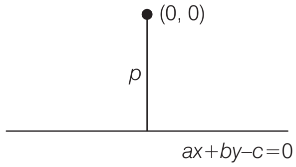
Which of the above is/are correct?

(a) I, II and III (b) I only
(c) I and II (d) II only

(c) We know that x_1 distance of a point $(x_1\ y_1)$ from the line $Ax + By + C = 0$ is given as

$$\text{Distance} = \left[\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}\right]$$

Statement I



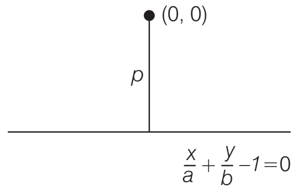
$$\therefore \quad p = \left|\frac{a \cdot 0 + b \cdot 0 - c}{\sqrt{a^2 + b^2}}\right|$$

$$\Rightarrow \quad p = \frac{c}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \quad p^2 = \frac{c^2}{a^2 + b^2}$$

It is true.

Statement II



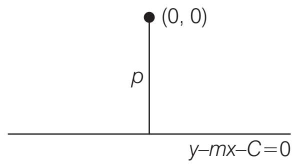
$$\therefore \quad p = \left|\frac{\dfrac{0}{a} + \dfrac{0}{b} - 1}{\sqrt{\left(\dfrac{1}{a}\right)^2 + \left(\dfrac{1}{b}\right)^2}}\right|$$

$$\Rightarrow \quad p = \frac{1}{\sqrt{\dfrac{1}{a^2} + \dfrac{1}{b^2}}}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

It is true.

Statement III



$$\therefore \quad p = \left|\frac{0 - m \times 0 - c}{\sqrt{(-m)^2 + (1)^2}}\right|$$

$$\Rightarrow \quad p = \frac{c}{\sqrt{m^2 + 1}}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1 + m^2}{c^2}$$

It is false.

56. What is the equation of the ellipse whose vertices are $(\pm 5, 0)$ and foci are at $(\pm 4, 0)$?

(a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

(b) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(c) $\frac{x^2}{25} + \frac{y^2}{16} = 1$

(d) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

(a) We have,

Vertices = $(\pm 5, 0)$ and Foci = $(\pm 4, 0)$

$\therefore$ $a = 5$ and $ae = 4$

[$\because$ vertex = $(\pm a, 0)$ and focus $(\pm ae, 0)$]

$\Rightarrow e = \frac{4}{5}$

Now, $e = \sqrt{1 - \frac{b^2}{a^2}}$

$\Rightarrow \left(\frac{4}{5}\right)^2 = 1 - \frac{b^2}{(5)^2}$ [$\because a = 5$]

$\Rightarrow \frac{16}{25} = 1 - \frac{b^2}{25}$

$\Rightarrow 16 = 25 - b^2$

$\Rightarrow b^2 = 9$

$\Rightarrow b = 3$

$\therefore$ Equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$$

57. What is the equation of the straight line passing through the point (2, 3) and making an intercept on the positive Y-axis equal to twice its intercept on the positive X-axis?

(a) $2x + y = 5$ (b) $2x + y = 7$
(c) $x + 2y = 7$ (d) $2x - y = 1$

(b) Let the equation of line be

$$\frac{x^2}{a} + \frac{y}{b} = 1$$

It is given that, $b = 2a$ and line passes through the point (2, 3).

$\therefore \frac{2}{a} + \frac{3}{2a} = 1$

$\Rightarrow \frac{4 + 3}{2a} = 1$

$\Rightarrow 7 = 2a$

$\Rightarrow a = \frac{7}{2}$

$\Rightarrow b = 2a = 2 \times \frac{7}{2} = 7$

$\therefore$ Equation of line is $\frac{x}{7/2} + \frac{y}{7} = 1$

$\Rightarrow 2x + y = 7$

58. Let the coordinates of the points A, B, C be $(1, 8, 4), (0, -11, 4)$ and $(2, -3, 1)$ respectively. What are the coordinates of the point D which is the foot of the perpendicular from A on BC?

(a) (3, 4, – 2) (b) (4, – 2, 5)
(c) (4, 5, – 2) (d) (2, 4, 5)

(c) We have,

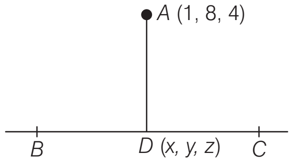
A (1, 8, 4), B (0, – 11, 4) and C (2, – 3, 1)

$\therefore$ Equation of BC is

$$\frac{x - 0}{2 - 0} = \frac{y + 11}{-3 + 11} = \frac{z - 4}{1 - 4}$$

$\Rightarrow \frac{x}{2} = \frac{y + 11}{8} = \frac{z - 4}{-3} = \lambda$ [say]

$\Rightarrow x = 2\lambda, y = 8\lambda - 11, z = -3\lambda + 4$



Now, DR's of

$AD = < 2\lambda - 1, 8\lambda - 11 - 8, -3\lambda + 4 - 4 >$

$= < 2\lambda - 1, 8\lambda - 19, -3\lambda >$

Since, $AD \perp BC$

$\therefore 2(2\lambda - 1) + 8(8\lambda - 19) - 3(-3\lambda) = 0$

$\Rightarrow 4\lambda - 2 + 64\lambda - 152 + 9\lambda = 0$

$\Rightarrow 77\lambda = 154$

$\Rightarrow \lambda = 2$

$\therefore$ Coordinates of

$D = (2 \times 2, 8 \times 2 - 11, -3 \times 2 + 4)$
$= (4, 5, -2)$

59. What is the equation of the plane passing through the points $(-2, 6, -6)$, $(-3, 10, -9)$ and $(-5, 0, -6)$?

(a) $2x - y - 2z = 2$
(b) $2x + y + 3z = 3$
(c) $x + y + z = 6$
(d) $x - y - z = 3$

(a) Equation of the plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Equation of plane is

$$\begin{vmatrix} x - (-2) & y - 6 & z - (-6) \\ -3 - (-2) & 10 - 6 & -9 - (-6) \\ -5 - (-2) & 0 - 6 & -6 - (-6) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x + 2 & y - 6 & z + 6 \\ -1 & 4 & -3 \\ -3 & -6 & 0 \end{vmatrix} = 0$$

$\Rightarrow (x + 2)[4 \times 0 - (-6)(-3)] - (y - 6)$
$[(-1)(0) - (-3)(-3)] + (z + 6)[(-1)(-6)$
$-(-3)(4)] = 0$

$\Rightarrow (x + 2)(-18) - (y - 6)(-9) +$
$(z + 6)(18) = 0$

$\Rightarrow 2(x + 2) - (y - 6) - 2(z + 6) = 0$

$\Rightarrow 2x - y - 2z - 2 = 0$

$\Rightarrow 2x - y - 2z = 2$

60. A sphere of constant radius r through the origin intersects the coordinate axes in A, B and C. What is the locus of the centroid of the ΔABC?

(a) $x^2 + y^2 + z^2 = r^2$
(b) $x^2 + y^2 + z^2 = 4r^2$
(c) $9(x^2 + y^2 + z^2) = 4r^2$
(d) $3(x^2 + y^2 + z^2) = 2r^2$

(c) Let $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$

$\therefore$ Equation of sphere passing through A, B, C and origin is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

$\therefore$ Radius = $r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}}$

$\Rightarrow 4r^2 = a^2 + b^2 + c^2$ ... (i)

Let (α, β, γ) be the centroid of triangle.

$\therefore \alpha = \frac{a + 0 + 0}{3}, \beta = \frac{0 + b + 0}{3}$,

$\gamma = \frac{0 + 0 + c}{3}$

$\Rightarrow a = 3\alpha, \beta = 3\beta, c + 3\gamma$ ... (ii)

From Eqs. (i) and (ii), we have

$(3\alpha)^2 + (3\beta)^2 + (3\gamma)^2 = 4r^2$

$\Rightarrow 9(\alpha^2 + \beta^2 + \gamma^2) = 4r^2$

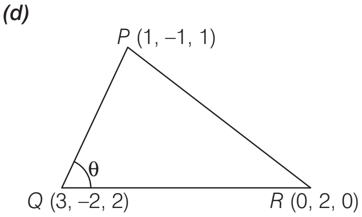
$\therefore$ Locus of the centroid of ΔABC is

$9(x^2 + y^2 + z^2) = 4r^2z$

61. The coordinates of the vertices P, Q and R of a triangle PQR are $(1, -1, 1), (3, -2, 2)$ and $(0, 2, 6)$ respectively. If $\angle RQP = 9$, then what is $\angle PRQ$ equal to ?

(a) $30° + \theta$ (b) $45° - \theta$
(c) $60° - \theta$ (d) $90° - \theta$

(d)



DR's of

$PQ = < 3 - 1, -2 - (-1), 2 - 1 >$

$< a_1, b_1, c_1 > = < 2, -1, 1 >$

and DR's of

$PR = < 0 - 1, 2 - (-1), 6 - 1 >$

$< a_2, b_2, c_2 > = < -1, 3, 5 >$

Now, $a_1a_2 + b_1b_2 + c_1c_2 =$
$2 \times (-1) + (-1) \times 3 + 1 \times 5$

$= -2 - 3 + 5 = 0$

$\therefore\ PQ \perp PR$

$\Rightarrow\ \angle QPR = 90^\circ$

Now, by angle sum property

$\angle PQR + \angle QPR + \angle PRQ = 180^\circ$

$\Rightarrow\quad \theta + 90^\circ + \angle PRQ = 180^\circ$

$\Rightarrow\quad \angle PRQ = 90^\circ - \theta$

62. The perpendiculars that fall from any point of the straight line $2x + 11y = 5$ upon the two straight lines $24x + 7y = 20$ and $4x - 3y = 2$ are

(a) 12 and 4 respectively
(b) 11 and 5 respectively
(c) Equal to each other
(d) Not equal to each other

(c) Let $(-3, 1)$ be a point on $2x + 11y = 5$

Now, perpendicular from $(-3, 1)$ on $24x + 7y = 20$

$$= \left|\frac{24(-3) + 7(1) - 20}{\sqrt{(24)^2 + (7)^2}}\right|$$

$$= \left|\frac{-72 + 7 - 20}{\sqrt{576 + 49}}\right|$$

$$= \left|\frac{-85}{25}\right| = \frac{17}{5}$$

Again, perpendicular from $(-3, 1)$ on $4x - 3y = 2$

$$= \left|\frac{4(-3) - 3(1) - 2}{\sqrt{4^2 + (-3)^2}}\right|$$

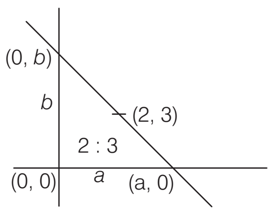
$$= \left|\frac{-12 - 3 - 2}{\sqrt{16 + 9}}\right| = \frac{17}{5}$$

$\therefore$ Both perpendicular are equal to each other.

63. The equation of the line, when the portion of it intercepted between the axes is divided by the point (2, 3) in the ratio of 3 : 2, is

(a) Either $x + y = 4$ or $9x + y = 12$
(b) Either $x + y = 5$ or $4x + 9y = 30$
(c) Either $x + y = 4$ or $x + 9y = 12$
(d) Either $x + y = 5$ or $9x + 4y = 30$

(d) ***Case* I**



From above figure,

$$\frac{2a + 3 \times 0}{2 + 3} = 2 \text{ and } \frac{2 \times 0 + 3b}{2 + 3} = 3$$

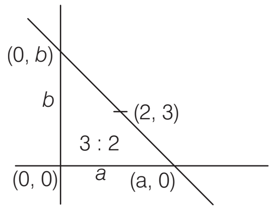
$\Rightarrow\quad 2a = 10$ and $3b = 15$

$\Rightarrow\quad a = 5$ and $b = 5$

$\therefore$ Equation of line is $\frac{x}{5} + \frac{y}{5} = 1$

$\Rightarrow\quad x + y = 5$

Case **II**



From above figure,

$$\frac{3a + 2 \times 0}{3 + 2} = 2 \text{ and } \frac{3 \times 0 + 2b}{3 + 2} = 3$$

$\Rightarrow\quad 3a = 10$ and $2b = 15$

$\Rightarrow\quad a = \frac{10}{3}$ and $b = \frac{15}{2}$

$\therefore$ Equation of line is

$$\frac{x}{10/3} + \frac{y}{15/2} = 1$$

$\Rightarrow\quad 9x + 4y = 30$

64. What is the distance between the straight lines $3x + 4y = 9$ and $6x + 8y = 15$?

(a) $\frac{3}{2}$ (b) $\frac{3}{10}$
(c) 6 (d) 5

(b) Given equation of straight lines are

$3x + 4y = 9$ … (i)

and $6x + 8y = 15$

$\Rightarrow\quad 3x + 4y = \frac{15}{2}$ … (ii)

$$\therefore \text{ Required distance} = \frac{\left|9 - \frac{15}{2}\right|}{\sqrt{3^2 + 4^2}}$$

[$\because$ distance between two lines $ax + by = c_1$ and $ax + by = c_2$ is given by $\left|\frac{c_2 - c_1}{\sqrt{a^2 + b^2}}\right|$]

$$= \frac{\frac{3}{2}}{5} = \frac{3}{10}$$

65. What is the equation of the sphere whose centre is at $(-2, 3, 4)$ and radius is 6 units?

(a) $x^2 + y^2 + z^2 + 4x - 6y - 8z = 7$
(b) $x^2 + y^2 + z^2 + 6x - 4y - 8z = 7$
(c) $x^2 + y^2 + z^2 + 4x - 6y - 8z = 4$
(d) $x^2 + y^2 + z^2 + 4x + 6y + 8z = 4$

(a) Given, centre = $(-2, 3, 4)$ and radius = 6 units

Equation of the sphere having centre at (α, β, γ) and radius r is $(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2$

So, equation of sphere

$\Rightarrow\quad \{x - (-2)\}^2 + (y - 3)^2 + (z - 4)^2 = 6^2$

$\Rightarrow\ (x + 2)^2 + (y - 3)^2 + (z - 4)^2 = 36$

$\Rightarrow\quad x^2 + 4x + 4 + y^2 + 9 - 6y + z^2 + 16 - 8z = 36$

$\Rightarrow\ x^2 + y^2 + z^2 + 4x - 6y - 8z + 29 - 36 = 0$

$\Rightarrow\quad x^2 + y^2 + z^2 + 4x - 6y - 8z = 7$

66. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $\vec{a} \times \vec{b} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$, then what is the acute angle between **a** and **b**?

(a) 30° (b) 45° (c) 60° (d) 90°

(a) Given, $|\vec{\mathbf{a}}| = 2$

$|\vec{\mathbf{b}}| = 7$

and $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

$\because\quad |\vec{\mathbf{a}} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}| \sin\theta$

$$\Rightarrow\quad \sin\theta = \frac{|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|}{|\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|}$$

$$= \frac{|3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}|}{2 \times 7}$$

$$= \frac{\sqrt{3^2 + 2^2 + 6^2}}{14} = \frac{\sqrt{49}}{14}$$

$\Rightarrow\ \sin\theta = \frac{7}{14} = \frac{1}{2}$

$\Rightarrow\ \sin\theta = \sin 30^\circ$

$\Rightarrow\quad \theta = 30^\circ$

67. Let $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ be the position vectors of the points P and Q respectively with respect to origin O. The points R and S divide PQ internally and externally respectively in the ratio 2 : 3. If $\vec{\mathbf{OR}}$ and $\vec{\mathbf{OS}}$ are perpendicular, then which one of the following is correct?

(a) $9p^2 = 4q^2$ (b) $4p^2 = 9q^2$
(c) $9p = 4q$ (d) $4p = 9q$

(a) The points R and S divider s PQ internally and externally respectively in the ratio 2 : 3. The position vectors of R and S are

$\frac{3\vec{\mathbf{p}} + 2\vec{\mathbf{q}}}{5}$ and $3\vec{\mathbf{p}} - 2\vec{\mathbf{q}}$ respectively.

$$\overrightarrow{OR} = \frac{3\vec{\mathbf{p}} - 2\vec{\mathbf{q}}}{5}$$

$$\overrightarrow{OS} = 3\vec{\mathbf{p}} - 2\vec{\mathbf{q}}$$

Now, $\overrightarrow{OR} \perp \overrightarrow{OS}$

$\Rightarrow\quad \overrightarrow{OR} \cdot \overrightarrow{OS} = 0$

$$\Rightarrow\quad \left(\frac{3\vec{\mathbf{p}} + 2\vec{\mathbf{q}}}{5}\right) \cdot (3\vec{\mathbf{p}} - 2\vec{\mathbf{q}}) = 0$$

$\Rightarrow\quad (3\vec{\mathbf{p}} + 2\vec{\mathbf{q}}) \cdot (3\vec{\mathbf{p}} - 2\vec{\mathbf{q}}) = 0$

$\Rightarrow \quad 9\vec{p}.\vec{p} - 6\vec{p}.\vec{q} + 6\vec{q}.\vec{p} - 4\vec{q}.\vec{q} = 0$

$\Rightarrow \quad 9|\vec{p}|^2 - 4|\vec{q}|^2 = 0$

$[\because \vec{a}.\vec{a} = |\vec{a}|^2 \text{ and } \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$

$\Rightarrow \quad 9|\vec{p}|^2 = 4|\vec{q}|^2$

$\Rightarrow \quad 9p^2 = 4q^2$

68. What is the moment about the point $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ of a force represented by $3\hat{\mathbf{i}} + \hat{\mathbf{k}}$ acting through the point $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$?

(a) $-3\hat{\mathbf{i}} + 11\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$ (b) $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$

(c) $3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

(a) Given that,

$\vec{r} = (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

$= \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$

and $\quad \vec{F} = 3\hat{\mathbf{i}} + \hat{\mathbf{k}}$

$\therefore$ Moment $\vec{\tau} = \vec{r} + \vec{F}$

$= (\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \times (3\hat{\mathbf{i}} + \hat{\mathbf{k}})$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -3 & 4 \\ 3 & 0 & 1 \end{vmatrix}$$

$= \hat{\mathbf{i}}(-3 - 0) - \hat{\mathbf{j}}(1 - 12) + \hat{\mathbf{k}}(0 + 9)$

$= -3\hat{\mathbf{i}} + 11\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$

69. If $\vec{\mathbf{a}} + 2\vec{\mathbf{b}} + 3\vec{\mathbf{c}} = \vec{\mathbf{0}}$ and $\vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}} + \vec{\mathbf{c}} \times \vec{\mathbf{a}} = \lambda\,(\vec{\mathbf{b}} \times \vec{\mathbf{c}})$, then what is the value of λ ?

(a) 2 (b) 3
(c) 4 (d) 6

(d) Given that,

$\vec{a} + 2\vec{b} + 3\vec{c} = 0$

$\Rightarrow \quad \vec{a} + 2\vec{b} = -3\vec{c}$

$\Rightarrow \quad (\vec{a} + 2\vec{b}) \times \vec{b} = -3\vec{c} \times \vec{b}$

$\Rightarrow \quad \vec{a} \times \vec{b} + 2\vec{b} \times \vec{b} = 3(\vec{b} \times \vec{c})$

$[\because \vec{c} \times \vec{b} = -\vec{b} \times \vec{c}]$

$\Rightarrow \quad \vec{a} \times \vec{b} = 3(\vec{b} \times \vec{c}) \quad [\vec{b} \times \vec{b} = 0]$ … (i)

Again $\quad 3\vec{c} + \vec{a} = -2\vec{b}$

$\Rightarrow \quad (3\vec{c} + \vec{a}) \times \vec{a} = -2\vec{b} \times \vec{a}$

$\Rightarrow \quad 3\vec{c} \times \vec{a} + \vec{a} \times \vec{a} = 2(\vec{a} \times \vec{b})$

$\Rightarrow \quad 3(\vec{c} \times \vec{a}) = 2(\vec{a} \times \vec{b})$

$\Rightarrow \quad 3(\vec{c} \times \vec{a}) = 6(\vec{b} \times \vec{c})$

[from Eq. (i)]

$\Rightarrow \quad \vec{c} \times \vec{a} = 2(\vec{b} \times \vec{c})$ … (ii)

Now, $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \lambda(\vec{b} \times \vec{c})$

$= 3(\vec{b} \times \vec{c}) + (\vec{b} \times \vec{c}) + 2(\vec{b} \times \vec{c})$

$= 6(\vec{b} \times \vec{c})$ [from Eqs. (i) and (ii)]

on comparing, we get

$\lambda = 6$

70. If the vectors $\vec{\mathbf{K}}$ and $\vec{\mathbf{A}}$ are parallel to each other, then what is $k\vec{\mathbf{K}} \times \vec{\mathbf{A}}$ equal to ?

(a) $k^2\vec{\mathbf{A}}$ (b) $\vec{\mathbf{0}}$ (c) $-k^2\vec{\mathbf{A}}$ (d) $\vec{\mathbf{A}}$

(b) Since, $\vec{a} \times \vec{b} = 0$, if $\vec{a}$ and $\vec{b}$ are parallel.

So, $k\,\vec{K} \times \vec{A} = 0$ if $\vec{K}$ and $\vec{A}$ are parallel to each other.

71. When one of the following is correct in respect of the function $f : \mathbf{R} \to \mathbf{R}^+$ defined as $f(x) = |x + 1|$?

(a) $f(x^2) = [f(x)]^2$
(b) $f(|x|) = |f(x)|$
(c) $f(x + y) = f(x) + f(y)$
(d) None of the above

(d) Given, $f(x) = |x + 1|$

By checking the options, we get

(a) $\quad f(x^2) = |x^2 + 1|$

$\{f(x)\}^2 = (x + 1)^2$

Which implies that $f(x^2) \neq \{f(x)\}^2$

(b) $\quad f(|x|) = ||x| + 1|$

$|f(x)| = ||x + 1|| = |x + 1|$

which implies that $f(|x|) \neq |f(x)|$

(c) $\quad f(x + y) = |x + y + 1|$

$f(x) + f(y) = |x + 1| + |y + 1|$

which implies that $f(x + y) \neq f(x) + f(y)$

So, option (d) is correct.

72. Suppose $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = \dfrac{x^2}{1 + x^2}$. What is the range of the function?

(a) [0, 1) (b) [0, 1] (c) (0, 1] (d) (0, 1)

(a) Let $f(x) = y$

Then, $y \geq 0$ and $f(x) = y$

$\therefore \quad \dfrac{x^2}{x^2 + 1} = y$

$\Rightarrow \quad \dfrac{x^2 + 1}{x^2} = \dfrac{1}{y}$ for $y > 0$

$\Rightarrow \quad \dfrac{1}{x^2} = \dfrac{1 - y}{y}$

$\Rightarrow \quad x = \sqrt{\dfrac{y}{1 - y}}$

Now, $\sqrt{\dfrac{y}{1 - y}}$ is real $\Rightarrow \dfrac{y}{1 - y} \geq 0$

$\Rightarrow 0 \leq y < 1$

So, Range of $f(x)$ is [0, 1).

73. If $f(x) = |x| + |x - 1|$, then which one of the following is correct?

(a) $f(x)$ is continuous at $x = 0$ at $x = 1$
(b) $f(x)$ is continuous at $x = 0$ but not at $x = 1$
(c) $f(x)$ is continuous at $x = 1$ but not at $x = 0$
(d) $f(x)$ is neither continuous at $x = 0$ nor at $x = 1$

(a) We have,

$f(x) = |x| + |x - 1|$

$$\Rightarrow \quad f(x) = \begin{cases} -2x + 1, & x < 0 \\ x - x + 1, & 0 \leq x < 1 \\ x + x - 1, & x \geq 1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -2x + 1, & x < 0 \\ 1, & 0 \leq x < 1 \\ 2x - 1, & x \geq 1 \end{cases}$$

Clearly, $\lim_{x \to 0^{-1}} f(x) = 1 = \lim_{x \to 0^+} f(x)$

and $\lim_{x \to 1^-} f(x) = \lim_{x - 1^+} f(x)$.

So, $f(x)$ is continuous at $x = 0, 1$.

74. Consider the function $f(x) = \begin{cases} x^2 \ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$ What is $f'(0)$ equal to ?

(a) 0 (b) 1
(c) –1 (d) It does not exist

(a) Given function is

$$f(x) = \begin{cases} x^2 \ln|x|, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\therefore \quad f'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$

$= \lim_{h \to 0} \dfrac{h^2 \log h}{h}$

$= \lim_{h \to 0} h \log h = 0$

75. What is the area of the region bounded by the parabolas $y^2 = 6(x - 1)$ and $y^2 = 3x$?

(a) $\dfrac{\sqrt{6}}{3}$ (b) $\dfrac{2\sqrt{6}}{3}$ (c) $\dfrac{4\sqrt{6}}{3}$ (d) $\dfrac{5\sqrt{6}}{3}$

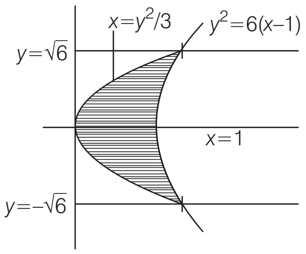
(c) Given,

$y^2 = 6(x - 1)$ … (i)

and $\quad y^2 = 3x$ … (ii)

on solving Eqs. (i) and (ii), we get

$x = 2$ and $y = \pm\sqrt{6}$



$\therefore$ Required area $= \int_{-\sqrt{6}}^{\sqrt{6}} \left(1 + \frac{y^2}{6} - \frac{y^2}{3}\right) dy$

$= 2\int_0^{\sqrt{6}} \left(1 - \frac{y^2}{6}\right) dy$

$= 2\left[y - \frac{y^3}{18}\right]_0^{\sqrt{6}}$

$= 2 \times \left[\frac{18y - y^3}{18}\right]_0^{\sqrt{6}}$

$= 2 \times \left[\frac{18\sqrt{6} - 6\sqrt{6}}{18}\right]$

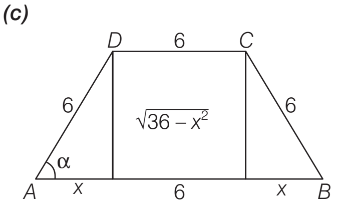
$= \frac{12\sqrt{6}}{9} = \frac{4\sqrt{6}}{3}$

Directions (Q. Nos. 76-78) *Consider the following information for the next three items that follow Three sides of a trapezium are each equal to 6 cm. Let $\alpha \varepsilon \left(0, \frac{\pi}{2}\right)$ be the angle between a pair of adjacent sides.*

76. If the area of the trapezium is the maximum possible, then what is α equal to ?

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{5}$

(c)



$\therefore$ Area $= A = \frac{1}{2}(6 + 6 + 2x)\sqrt{36 - x^2}$

$= (6 + x)\sqrt{36 - x^2}$

$\therefore \frac{d(A)}{dx} = \frac{d}{dx}[(6 + x)\sqrt{36 - x^2}]$

$= (6 + x)\left[\frac{-2x}{2\sqrt{36 - x^2}}\right] + \sqrt{36 - x^2}$

$= \sqrt{36 - x^2} - \frac{x(6 + x)}{\sqrt{36 - x^2}}$

$= \frac{36 - 6x - 2x^2}{\sqrt{36 - x^2}}$

For maximum area,

$\frac{dA}{dx} = 0$

$\Rightarrow \quad 36 - 6x - 2x^2 = 0$

$\Rightarrow \quad 2x^2 + 6x - 36 = 0$

$\Rightarrow \quad x^2 + 3x - 18 = 0$

$\Rightarrow \quad x^2 + 6x - 3x - 18 = 0$

$\Rightarrow \quad x(x + 6) - 3(x + 6) = 0$

$\Rightarrow \quad (x + 6)(x - 3) = 0$

$\Rightarrow \quad x = 3, -6$

Again, on differentiating it

$\frac{d^2A}{dx^2} = \frac{(-6 - 4x)\sqrt{36 - x^2} - \frac{(36 - 6x - 2x^2)(-2x)}{2\sqrt{36 - x^2}}}{(36 - x^2)}$

At $x = 3$, $\frac{d^2A}{dx^2} = -6 - 12 = -18$

$\therefore \quad \frac{d^2A}{dx^2} < 0$

So, at $x = 3$ is maximum.

Now, $\cos\alpha = \frac{x}{6} = \frac{3}{6} = \frac{1}{2}$

$\Rightarrow \quad \cos\alpha = \cos\frac{\pi}{3} \Rightarrow \alpha = \frac{\pi}{3}$

77. If the area of the trapezium is maximum, what is the length of the fourth side?

(a) 8 cm (b) 9 cm
(c) 10 cm (d) 12 cm

(d) So, fourth side $= x + 6 + x$

$= 3 + 6 + 3 = 12$

78. What is the maximum area of the trapezium?

(a) $36\sqrt{3}$ cm^2 (b) $30\sqrt{3}$ cm^2
(c) $27\sqrt{3}$ cm^2 (d) $24\sqrt{3}$ cm^2

(c) Maximum area $= (6 + x)\sqrt{36 - x^2}$

$= (6 + 3)\sqrt{36 - 9}$

$= 9 \times \sqrt{27} = 9 \times \sqrt{27}$

$= 9 \times 3\sqrt{3} = 27\sqrt{3}$ cm^2

79. What is $\int_0^{\pi} e^x \sin x \, dx$ equal to ?

(a) $\frac{e^{\pi} + 1}{2}$ (b) $\frac{e^{\pi} - 1}{2}$
(c) $e^{\pi} + 1$ (d) $\frac{e^{\pi} + 1}{4}$

(a) Let $I = \int_0^{\pi} e^x \sin x \, dx$

$= [\sin x \cdot e^x]_0^{\pi} - \int_0^{\pi}\left[\frac{d}{dx}\{\sin x\} \cdot e^x\right] dx$

$= [\sin x \cdot e^x]_0^{\pi} - \int_0^{\pi} \cos x \cdot e^x dx$

$= 0 - \left\{[\cos x \cdot e^x]_0^{\pi} + \int_0^{\pi} \sin x \cdot e^x dx\right\}$

$\Rightarrow \quad I = -[-e^{\pi} - 1] - I$

$\Rightarrow \quad I + I = e^{\pi} + 1$

$\Rightarrow \quad 2I = e^{\pi} + 1$

$\Rightarrow \quad I = \frac{e^{\pi} + 1}{2}$

80. If $f(x) = \frac{x - 9}{x^2 - 2x - 3}$, $x \neq 3$ is continuous at $x = 3$, then which one of the following is correct?

(a) $f(3) = 0$ (b) $f(3) = 1.5$
(c) $f(3) = 3$ (d) $f(3) = -1.5$

(b) Since, $f(x)$ is continuous at $x = 3$

Therefore, $f(3) = \lim_{x \to 3} \frac{x^2 - 9}{x^2 - 2x - 3}$

Applying L′ Hospital rule

$f(3) = \lim_{x \to 3} \frac{\frac{d}{dx}(x^2 - 9)}{\frac{d}{dx}(x^2 - 2x - 3)}$

$= \lim_{x \to 3} \frac{2x}{2x - 2}$

$= \frac{2 \cdot 3}{2 \cdot 3 - 2} = \frac{6}{4} = 1.5$

81. What is $\int_1^e x \ln x \, dx$ equal to ?

(a) $\frac{e + 1}{4}$ (b) $\frac{e^2 + 1}{4}$
(c) $\frac{e - 1}{4}$ (d) $\frac{e^2 - 1}{4}$

(b) Let $1 = \int_1^e x \log x dx$

$= [\log x \cdot \int x dx]_1^e - \int_1^e \left[\frac{d}{dx}\{\log x\} \cdot \int x dx\right] dx$

$= \left[\log x \cdot \frac{x^2}{2}\right]_1^e - \int_1^e \frac{1}{x} \cdot \frac{x^2}{2} dx$

$= \frac{e^2}{2} - \frac{1}{2} \times \frac{1}{2}[x^2]_1^e$

$= \frac{e^2}{2} - \frac{[e^2 - 1]}{4}$

$= \frac{2e^2 - e^2 + 1}{4} = \frac{e^2 + 1}{4}$

82. What is $\int_0^{\sqrt{2}} [x^2] \, dx$ equal to (where [.] is the greatest integer function) ?

(a) $\sqrt{2} - 1$ (b) $1 - \sqrt{2}$
(c) $2(\sqrt{2} - 1)$ (d) $\sqrt{3} - 1$

(a) Let $I = \int_0^{\sqrt{2}} [x^2] dx$

$= \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx$

$= \int_0^1 0 \, dx + \int_1^{\sqrt{2}} 1 \, dx$

$\left[\because [x] = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < \sqrt{2} \end{cases}\right]$

$= 0 + [x]_1^{\sqrt{2}} = \sqrt{2} - 1$

83. What is the maximum value of $16\sin\theta - 12\sin^2\theta$?

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{16}{3}$ (d) 4

(c) Let $f(x) = 16\sin\theta - 12\sin^2\theta$

$$= -12\left[\sin^2\theta - \frac{16}{12}\sin\theta\right]$$
$$= -12\left[\sin^2\theta - \frac{4}{3}\sin\theta\right]$$
$$= -12\left[\left(\sin\theta - \frac{2}{3}\right)^2 - \frac{4}{9}\right]$$
$$= -12\left(\sin\theta - \frac{2}{3}\right)^2 + \frac{16}{3}$$

$\therefore$ $f(x) \le \frac{16}{3}$

$\therefore$ Maximum value of $f(x) = \frac{16}{3}$

84. If $f : \mathbf{R} \to S$ defined by $f(x) = 4\sin x - 3\cos x + 1$ is onto, then what is S equal to ?

(a) $[-5, 5]$ (b) $(-5, 5)$
(c) $(-4, 6)$ (d) $[-4, 6]$

(d) We have

$$f(x) = 4\sin x - 3\cos x + 1$$

We know that,

$$-\sqrt{4^2 + (-3)^2} \le 4\sin x - 3\cos x \le \sqrt{4^2 + (-3)^2}$$

$$[\because -\sqrt{a^2 + b^2} \le a\sin x + b\cos x \le \sqrt{a^2 + b^2}]$$

$\Rightarrow -5 \le 4\sin x - 3\cos x \le 5$

$\Rightarrow -5 + 1 \le 4\sin x - 3\cos x + 1 \le 5 + 1$

$\Rightarrow -4 \le f(x) \le 6$

$\therefore$ $f(x) \in [-4, 6]$

since, $f(x)$ is onto.

$\therefore$ S = Range of $f = [-4, 6]$

85. For f to be a function, what is the domain of f, if $f(x) = \frac{1}{\sqrt{|x| - x}}$?

(a) $(-\infty, 0)$ (b) $(0, \infty)$
(c) $(-\infty, \infty)$ (d) $(-\infty, 0)$

(a) We have,

$$f(x) = \frac{1}{\sqrt{|x| - x}}$$

$f(x)$ is defined, if

$|x| - x > 0$

$\Rightarrow$ $|x| > x$

Case I $x > 0$

$\therefore$ $x > x$ $[\because |x| = x, x > 0]$

which is not possible

Case II $x < 0$

$\therefore$ $-x > x$ $[\because |x| = -x, x < 0]$

$\Rightarrow$ $0 > 2x$

$\Rightarrow$ $x < 0$

Which is possible

$\therefore$ Domain of $f(x) = (-\infty, 0)$

86. What is the solution of the differential equation $x\,dy - y\,dx = 0$?

(a) $xy = c$
(b) $y = cx$
(c) $x + y = c$
(d) $x - y = c$

(b) Given differentiation equation

$$x dy - y dx = 0$$

$\Rightarrow$ $x dy = y\,dx$

Variable seprate on both sides

$\Rightarrow$ $\frac{dy}{y} = \frac{dx}{x}$

On integration both sides, we get

$\Rightarrow$ $\int \frac{dy}{y} = \int \frac{dx}{x}$

$\Rightarrow$ $\log y = \log x + \log c$

[where $\log c$ is integrating constant]

$\Rightarrow$ $y = xc$

$\Rightarrow$ $y = cx$

87. What is the derivative of the function $f(x) = e^{\tan x} + \ln(\sec x) - e^{\ln x}$ at $x = \frac{\pi}{4}$?

(a) $\frac{e}{2}$ (b) e (c) $2e$ (d) $4e$

(c) We have,

$F(x) = e^{\tan x} + \log(\sec x) - e^{\log x}$

$f(x) = e^{\tan x} + \log(\sec x) - x$

$[\because a\log_a b = b]$

On differentiating with respect to x both the sides, we get

$$f'(x) = e^{\tan x} \cdot \sec^2 x + \frac{1}{\sec x} \cdot \sec x \tan x - 1$$
$$= \sec^2 x e^{\tan x} + \tan x - 1$$
$$\therefore [f'(x)]_{x=\frac{\pi}{4}} = e^{\tan\frac{\pi}{4}} \cdot \sec^2 \cdot \frac{\pi}{4} + \tan\frac{\pi}{4} - 1$$
$$= e^1(\sqrt{2})^2 + 1 - 1$$
$$= e \cdot 2 + 1 - 1 = 2e$$

88. Which one of the following differential equations has a periodic solution?

(a) $\frac{d^2x}{dt^2} + \mu x = 0$ (b) $\frac{d^2x}{dt^2} - \mu x = 0$

(c) $x\frac{dx}{dt} + \mu t = 0$ (d) $\frac{dx}{dt} + \mu xt = 0$

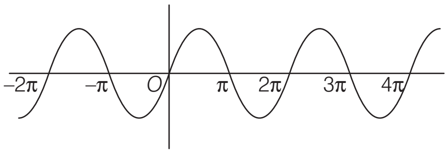
(a) $\frac{d^2x}{dt^2} + \mu x = 0$ is the differential equation of simple harmonic motion, which has a periodic solution.

89. What is the period of the the function $f(x) = \sin x$?

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) 2π

(d) We have,

$f(x) = \sin x$



$$f(x + 2\pi) = \sin(x + 2\pi).$$
$$= \sin x$$
$$= f(x)$$

$\therefore$ Period of $f(x)$ is 2π.

90. What is $\int \frac{dx}{2^x - 1}$ equal to ?

(a) $\ln(2^x - 1) + C$

(b) $\frac{\ln(1 - 2^{-x})}{\ln 2} + C$

(c) $\frac{\ln(2^{-x} - 1)}{2\ln 2} + C$

(d) $\frac{\ln(1 + 2^{-x})}{\ln 2} + C$

(b) Let $I = \int \frac{dx}{2x^2 - 1}$

$$= \int \frac{2^{-x}}{1 - 2^{-x}} dx$$
$$= \frac{1}{\log 2} \int \frac{2^{-x}\log 2}{1 - 2^{-x}} dx$$

Put $1 - 2^{-x} = t$

$\Rightarrow$ $2^{-x}\log 2\, dx = dt$

$\therefore$ $I = \frac{1}{\log 2} \int \frac{dt}{t}$

$$= \frac{1}{\log 2} . \log t + C$$
$$= \frac{\log(1 - 2^{-x})}{\log 2} + C$$

91. The order and degree of the differential equation $y^2 = 4a(x - a)$, where 'a' is an arbitrary constant, are respectively

(a) 1, 2 (b) 2, 1
(c) 2, 2 (d) 1, 1

(a) We have,

$y^2 = 4a(x - a)$ ... (i)

$\Rightarrow$ $y^2 = 4ax - 4a^2$

On differentiating both sides, we get

$$2y\frac{dy}{dx} = 4a$$

$\Rightarrow$ $a = \frac{1}{2}y\frac{dy}{dx}$ ... (ii)

On putting the values of a from Eq. (ii) in Eq. (i), we get

$$y^2 = 4 \times \frac{1}{2} y \frac{dy}{dx}\left(x - \frac{1}{2} y \frac{dy}{dx}\right)$$

$$\Rightarrow \quad y^2 = 2xy \frac{dy}{dx} - y^2\left(\frac{dy}{dx}\right)^2$$

∴ Order = 1 and degree = 2

92. What is the value of $\int_{-\pi/4}^{\pi/4} (\sin x - \tan x)\, dx$?

(a) $-\frac{1}{\sqrt{2}} + \ln\left(\frac{1}{\sqrt{2}}\right)$ (b) $\frac{1}{\sqrt{2}}$

(c) 0 (d) $\sqrt{2}$

(c) Let $I = \int_{-\pi/4}^{\pi/4} (\sin x - \tan x)\, dx$

Let $f(x) = \sin x - \tan x$

$\therefore \quad f(-x) = \sin(-x) - \tan(-x)$

$= -\sin x + \tan x$

$[\because \sin(-\theta) = -\sin\theta,\ \tan(-\theta) = -\tan\theta]$

$= -(\sin x - \tan x)$

$= -f(x)$

∴ $f(x)$ is odd function.

$\therefore \quad I = \int_{-\pi/4}^{\pi/4} (\sin x - \tan x)\, dx$

$= 0$

$[\because \int_{-a}^{a} f(x)\, dx = 0, \text{ if } f(x) \text{ is odd}]$

93. If $\int_a^b x^3\, dx = 0$ and $\int_a^b x^2\, dx = \frac{2}{3}$, then what are the values of a and b respectively?

(a) – 1, 1 (b) 1, 1

(c) 0, 0 (d) 2, –2

(a) We have,

$$\int_a^b x^3 dx = 0$$

$$\Rightarrow \left[\frac{x^4}{4}\right]_a^b = 0$$

$$\Rightarrow \frac{b^4 - a^4}{4} = 0$$

$$\Rightarrow b^4 = a^4$$

$$\Rightarrow b = \pm a \quad \ldots \text{(i)}$$

But $\int_a^b x^3\, dx = 0$ and x^3 is an odd function.

$$\therefore \quad a = -b \quad \ldots \text{(ii)}$$

Again, $\int_a^b x^2 dx = \frac{2}{3}$

$$\Rightarrow \left[\frac{x^3}{3}\right]_a^b = \frac{2}{3}$$

$$\Rightarrow \frac{b^3 - a^3}{3} = \frac{2}{3}$$

$$\Rightarrow b^3 - a^3 = 2$$

$$\Rightarrow b^3 - (-b)^3 = 2 \quad [\because \text{from Eq. (ii)}]$$

$$\Rightarrow 2b^3 = 2$$

$$\Rightarrow b^3 = 1$$

$$\Rightarrow b = 1$$

$$\Rightarrow a = -1 \quad \text{[from Eq. (ii)]}$$

94. What is $\int_0^1 x(1-x)^9\, dx$ equal to ?

(a) $\frac{1}{110}$ (b) $\frac{1}{132}$

(c) $\frac{1}{148}$ (d) $\frac{1}{240}$

(a) Let $I = \int_0^1 x(1-x)^9\, dx$

$$= \int_0^1 (1-x)\, x^9\, dx$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^1 (x^9 - x^{10})\, dx = \left[\frac{x^{10}}{10} - \frac{x^{11}}{11}\right]_0^1$$

$$= \left(\frac{1}{10} - \frac{1}{11}\right) = \frac{1}{110}$$

95. What is $\lim_{x \to 0} \frac{\tan x}{\sin 2x}$ equal to

(a) $\frac{1}{2}$

(b) 1

(c) 2

(d) Limit does not exist

(a) We have,

$$\lim_{x \to 0} \frac{\tan x}{\sin 2x}$$

By using L' Hospital rule, we have

$$= \lim_{x \to 0} \frac{\sec^2 x}{2\cos 2x}$$

$$= \lim_{x \to 0} \frac{1}{2\cos 2x \cdot \cos^2 x}$$

$$= \frac{1}{2\cos 0° \cos^2 0°} = \frac{1}{2 \times 1 \times 1}$$

$$= \frac{1}{2}$$

96. What is $\lim_{x \to 0} \frac{\sqrt{2x + 3h} - \sqrt{2x}}{2h}$ equal to ?

(a) $\frac{1}{2\sqrt{2x}}$ (b) $\frac{1}{\sqrt{2x}}$

(c) $\frac{3}{2\sqrt{2x}}$ (d) $\frac{3}{4\sqrt{2x}}$

(d) $\lim_{x \to 0} \frac{\sqrt{2x + 3h} - \sqrt{2x}}{2h}$

By using, L' Hospital rule, we get

$$= \lim_{h \to 0} \frac{\frac{1}{2\sqrt{2x + 3h}} \cdot 3 - 0}{2}$$

$$= \lim_{h \to 0} \frac{3}{4\sqrt{2x + 3h}}$$

$$= \frac{3}{4} \cdot \frac{1}{\sqrt{2x + 0}} = \frac{3}{4\sqrt{2x}}$$

97. If $f(x)$ is an even function, where $f(x) \neq 0$, then which one of the following is correct?

(a) $f'(x)$ is an even function

(b) $f'(x)$ is an odd function

(c) $f'(x)$ may be an even or odd function depending on the type of function

(d) $f'(x)$ is a constant function

(b) We have, $f(x)$ is an even function.

$\therefore \quad f(-x) = f(x)$

On differentiating both the sides, we have

$$-f'(-x) = f'(x)$$

$$\Rightarrow \quad f'(-x) = -f'(x)$$

∴ $f'(x)$ is an odd function.

98. If $y = e^{x^2} \sin 2x$, then what is $\frac{dy}{dx}$ at $x = \pi$ equal to ?

(a) $(1 + \pi)e^{\pi^2}$ (b) $2\pi e^{\pi^2}$

(c) $2e^{\pi^2}$ (d) e^{π^2}

(c) We have, $y = e^{x^2} \sin 2x$

On differentiating both the sides, we get

$$\frac{dy}{dx} = 2\cos 2x e^{x^2} + 2xe^{x^2} \sin 2x$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\pi} = 2\cos 2\pi e^{\pi^2} + 2\pi e^{\pi^2} \sin 2\pi$$

$$= 2(1)e^{\pi^2} + 2\pi e^{\pi^2}(0)$$

$$= 2e^{\pi^2}$$

99. What is the solution of $(1 + 2x)dy - (1 - 2y)dx = 0$?

(a) $x - y - 2xy = c$

(b) $y - x - 2xy = c$

(c) $y + x - 2xy = c$

(d) $x + y + 2xy = c$

(a) We have,

$$(1 + 2x)\, dy - (1 - 2y)\, dx = 0$$

$$\Rightarrow (1 + 2x)\, dy = (1 - 2y)\, dx$$

$$\Rightarrow \frac{dy}{1 - 2y} = \frac{dx}{1 + 2x}$$

On integrating both the sides, we get

$$\int \frac{dy}{1 - 2y} = \int \frac{dy}{1 + 2x}$$

$$\Rightarrow -\frac{1}{2}\log(1 - 2y) = \frac{1}{2}\log(1 + 2x) + C'$$

$$\Rightarrow -\log(1 - 2y) = \log(1 + 2x) + 2C'$$

$$\Rightarrow \log(1 + 2x) + \log(1 - 2y) = -2C'$$

$$\Rightarrow \log(1 + 2x)(1 - 2y) = -2C'$$

$$\Rightarrow (1 + 2x)(1 - 2y) = e^{-2C'}$$

$$\Rightarrow 1 - 2y + 2x - 4xy = e^{-2C'}$$

$$\Rightarrow 2x - 2y - 4xy = e^{-2C'} - 1$$

$\Rightarrow\ x - y - 2xy = \frac{1}{2}(e^{-2C'} - 1)$

$\Rightarrow\ x - y - 2xy = C$

[where $C = \frac{1}{2}(e^{-2C'} - 1)$]

100. What are the order and degree, respectively, of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2 = y^4 + \left(\frac{dy}{dx}\right)^5$?

(a) 4, 5 (b) 2, 3
(c) 3, 2 (d) 5, 4

(c) Given differential equation is

$$\left(\frac{d^3y}{dx^3}\right)^2 = y^4 + \left(\frac{dy}{dx}\right)^5$$

Here, highest order derivative is $\frac{d^3y}{dx^3}$.

So, order = 3, and degree = 2

101. In a Binomial distribution, the mean is three times its variance. What is the probability of exactly 3 successes out of 5 trials ?

(a) $\frac{80}{243}$ (b) $\frac{40}{243}$
(c) $\frac{20}{243}$ (d) $\frac{10}{243}$

(a) According to the question,

Mean = 3 (Variance)

$\Rightarrow\ np = 3npq$ [where n = number, of trials]

$\Rightarrow\ q = \frac{1}{3}$

$\because\ p + q = 1$

$\Rightarrow\ p + \frac{1}{3} = 1$

$\Rightarrow\ p = \frac{2}{3}$

$\therefore\ p(X = 3) = {}^5C_3\left(\frac{2}{3}\right)^3 \times \left(\frac{1}{3}\right)^2$

$= \frac{5!}{3!\,2!} \times \left(\frac{2}{3}\right)^3 \times \left(\frac{1}{3}\right)^2 = \frac{80}{243}$

102. Consider the following statements

I. $P(\bar{A} \cup B) = P(\bar{A}) = P(B) - P(\bar{A} \cap B)$

II. $P(A \cap \bar{B}) = P(B) - P(A \cap B)$

III. $P(A \cap B) = P(B)\,P(A \mid B)$

Which of the above statements are correct?

(a) I and II (b) I and III
(c) II and III (d) I, II and III

(b) Here,

Statement I

$P(\bar{A} \cup B) = P(\bar{A}) + P(B) - P(\bar{A} \cap B)$ is correct.

Statement II

$P(A \cap \bar{B}) = P(B) - P(A \cap B)$ is wrong as $P(A \cap \bar{B}) = P(A) - P(A \cap B)$.

Statement III

$P(A \cap B) = P(B) \times P\left(\frac{A}{B}\right)$ is correct. [by conditional theorem]

Hence, Statements I and III are correct.

103. If the correlation coefficient between x and y is 0.6, covariance is 27 and variance of y is 25, then what is the variance of x?

(a) $\frac{9}{5}$ (b) $\frac{81}{25}$
(c) 9 (d) 81

(d) Given, $\sigma^2(y) = 25 \Rightarrow \sigma(y) = 5$

Correlation coefficient = $\frac{C \cdot V}{\sigma_x \cdot \sigma_y}$

$0.6 = \frac{27}{\sigma_x \times \sqrt{5}}$

$\Rightarrow\ \sigma_x = \frac{27}{0.6 \times 5} = \frac{27}{3} = 9$

$\therefore$ Variance of $x = \sigma^2(x) = (9)^2 = 81$

104. The probabilities that a student will solve Question A and Questions B are 0.4 and 0.5 respectively. What is the probability that he solves atleast one of the two questions?

(a) 0.6 (b) 0.7
(c) 0.8 (d) 0.9

(b) Given that,

$P(A) = 0.4$

and $P(B) = 0.5$

$\because\ P(A \cup B) = 1 - P(A' \cap B')$

$= 1 - [(1 - 0.4) \times (1 - 0.5)]$

$= 1 - (0.6)(0.5)$

$= 1 - (0.3) = 0.7$

105. Let $\bar{x}$ be the mean of $x_1, x_2, x_3, \ldots x_n$. If $x_i = a + cy_i$ for some constants a and c, then what will be the mean of $y_1, y_2, y_3, \ldots, y_n$?

(a) $a + c\,\bar{x}$ (b) $a - \frac{1}{c}\bar{x}$
(c) $\frac{1}{c}\,\bar{x} - a$ (d) $\frac{\bar{x} - a}{c}$

(d) Given that,

Mean of $x_1, x_2, x_3, \ldots\, x_n$ i.e. $x = \bar{x}$

Now, we have

$x_i = a + cy_i$

$\Rightarrow\ y_i = \frac{1}{c}(x - a)$

$\Rightarrow\ \bar{y} = \frac{1}{c}(\bar{x} - a)$

106. Consider the following statements

I. If the correlation coefficient $\mathbf{r}_{xy} = 0$, then the two lines of regression are parallel to each other.

II. If the correlation coefficient $\mathbf{r}_{xy} = 1$, then the two lines of regression are perpendicular to each other.

Which of the above statements is/are correct?

(a) I only
(b) II only
(c) Both I and II
(d) Neither I nor II

(d) According to correlation condition,

If correlation coefficient $r_{r \cdot y} = 0$, then lines of regression are parpendicular

And if $r_{ry} = 1$, then lines of regression are parallel.

So, both statements are wrong.

107. If $4x - 5y + 33 = 0$ and $20x - 9y = 107$ are two lines of regression, then what are the values of $\bar{x}$ and $\bar{y}$ respectively?

(a) 12 and 18 (b) 18 and 12
(c) 13 and 17 (d) 17 and 13

(c) Given lines of regression are

$4x - 5y + 33 = 0$ … (i)

and $20x + 9y - 107 = 0$ … (ii)

on multiplying Eq. (i) by 5 and subtract Eq. (ii) from it, we get

$20x - 25y + 165 = 0$
$20x - 9y - 107 = 0$
$- \quad + \quad +$

$-16y = -272$

$y = 17$

on putting the value of y in Eq. (i), we get

$4x - 85 + 33 = 0$

$\Rightarrow\ 4x = 52 \Rightarrow x = 13$

The mean of two regression lines are the solution set at given regression lines,

Here, $\bar{X} = 13$ and $\bar{Y} = 17$

108. Consider the following statements

I. Mean is independent of change in scale and change in origin.

II. Variance is independent of change in scale but not in origin.

Which of the above statements is/are correct?

(a) I only (b) II only
(c) Both I and II (d) Neither I nor II

(d) Since, mean changes with changes in origin. So, Statement I is wrong.

And variance is independent to the choice of origin. So, Statement II is also wrong.

Hence, both statements are wrong.

109. Consider the following statements

I. The sum of deviations from mean is always zero.

II. The sum of absolute deviations is minimum when taken around median.

Which of the above statements is/are correct.

(a) I only (b) II only
(c) Both I and II (d) Neither I nor II

(c) By the property of deviation both statement are correct.

110. What is the median of the numbers 4.6, 0, 9.3, –4.8, 7.6 2.3, 12.7, 3.5, 8.2, 6.1, 3.9, 5.2 ?

(a) 3.8 (b) 4.9
(c) 5.7 (d) 6.0

(b) On arranging the given number is ascending order, we have

– 4.8, 0, 2.3, 3.5, 3.9, 4.6, 5.2, 6.1, 7.6, 8.2, 9.3, 12.7

Here, $n = 12$

So, median

$$= \frac{\text{Value of } \left(\frac{12}{2}\right) \text{th number} + \text{Value of} \left(\frac{12}{2}+1\right) \text{th number}}{2}$$

$$= \frac{\text{Value of 6th number} + \text{Value 7th number}}{2}$$

$$= \frac{4.6 + 52}{2}$$

$= 4.9$

111. In a test in Mathematics, 20% of the students obtained ''first class''. If the data are represented by a pie chart, what is the central angle corresponding to ''first class''?

(a) 20° (b) 36°
(c) 72° (d) 144°

(c) Pie chart contains total angle equal to 360°.

So, central angle corresponding to "First class"

$= 20\%$ of $360°$

$= \frac{20}{100} \times 360°$

$= 72°$

112. The mean and standard deviation of a set of values are 5 and 2 respectively. If 5 is added to each value, then what is the coefficient of variation for the new set of values?

(a) 10 (b) 20
(c) 40 (d) 70

(b) Given, mean = 5

and standard deviation $(\sigma) = 2$

Since, 5 is added to each value.

So, new mean = 5 + 5

= 10

But standard deviation will remain same.

Hence, coefficient of variation

$= \frac{\sigma}{\text{mean}} \times 100$

$= \frac{2}{10} \times 100 = 20$

113. A train covers the first 5 km of its journey at a speed of 30 km/h and the next 15 km at a speed of 45 km/h. What is the average speed of the train?

(a) 35 km/h (b) 37.5 km/h
(c) 39.5 km/h (d) 40 km/h

(d) Average speed $= \frac{\text{Total distance}}{\text{Total time}}$

$$= \frac{5 + 15}{\frac{5}{30} + \frac{15}{45}} = \frac{20}{\frac{1}{6} + \frac{1}{3}}$$

$$= \frac{20}{\frac{1+2}{6}}$$

$$= \frac{20 \times 6}{3} = 40 \text{ km/h}$$

114. Two fair dice are rolled. What is the probability of getting a sum of 7?

(a) $\frac{1}{36}$ (b) $\frac{1}{6}$
(c) $\frac{7}{12}$ (d) $\frac{5}{12}$

(b) Here, $n(S) = 36$

and E be the event of getting a sum of 7 on two fair dice.

$= \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$\therefore n(E) = 6$

So, required probability $= \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

115. If A and B are two events such that $2P(A) = 3P(B)$, where $0 < P(A) < P(B) < 1$, then which one of the following is correct?

(a) $P(A|B) < P(B|A) < P(A \cap B)$
(b) $P(A \cap B) < P(B|A) < P(A|B)$
(c) $P(B|A) < P(A|B) < P(A \cap B)$
(d) $P(A \cap B) < P(A|B) < P(B|A)$

(b) Given that, $2P(A) = 3P(B)$

$\Rightarrow 2\frac{P(A)}{P(A \cap B)} = \frac{3P(B)}{P(A \cap B)}$

[dividing both sides by $P(A \cup B)$]

$\Rightarrow \frac{1}{2} \times \frac{P(A \cap B)}{P(A)} = \frac{1}{3} \times \frac{P(A \cap B)}{P(B)}$

$\Rightarrow \frac{1}{2} \times P\left(\frac{B}{A}\right) = \frac{1}{3} P\left(\frac{A}{B}\right)$

$\Rightarrow P\left(\frac{B}{A}\right) < P\left(\frac{A}{B}\right)$

116. A box has ten chits numbered 0, 1, 2, 3, ..., 9. First, one chit is drawn at random and kept aside. From the remaining, a second chit is drawn at random. What is the probability that the second chit drawn is ''9'' ?

(a) $\frac{1}{10}$ (b) $\frac{1}{9}$
(c) $\frac{1}{90}$ (d) None of these

(a) Let E_1 be the event at drawing a chit which is not 9 and E_2 be the event of drawing second chit bearing number 9.

$\therefore \quad P(E_1) = \frac{{}^9C_1}{{}^{10}C_1} = \frac{9}{10}$

and $\quad P(E_2) = \frac{{}^1C_1}{{}^9C_1} = \frac{1}{9}$

$\therefore$ Required probability $= P(E_1) \cdot P(E_2)$

$= \frac{9}{10} \times \frac{1}{9} = \frac{1}{10}$

117. One bag contains 3 white and 2 black balls, another bag contains 5 white and 3 black balls. If a bag is chosen at random and a ball is drawn from it, what is the change that it is white?

(a) $\frac{3}{8}$ (b) $\frac{49}{80}$ (c) $\frac{8}{13}$ (d) $\frac{1}{2}$

(b) Let E_1 be the event of selecting the first bag and E_2 be the event of selecting the second bag. Let A be the event of drawing white ball.

So, by theorem at total probability.

$$P(A) = P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)$$

$$= \frac{1}{2} \times \frac{{}^3C_1}{{}^5C_1} + \frac{1}{2} \times \frac{{}^5C_1}{{}^8C_1}$$

$$= \frac{1}{2}\left[\frac{3}{5} + \frac{5}{8}\right] = \frac{1}{2} \times \frac{24 + 25}{40}$$

$$= \frac{1}{2} \times \frac{49}{40} = \frac{49}{80}$$

118. Consider the following in respect of two events A and B

I. $P(A$ occurs but not $B) = P(A) - P(B)$ if $B \subset A$

II. $P(A$ alone or B alone occurs$) = P(A) = P(B) - P(A \cap B)$

III. $P(A \cup B) = P(A) + P(B)$ if A and B are mutually exclusive

Which of the above is/are correct?

(a) I only (b) I and III
(c) II and III (d) I and II

(b) If $B \subset A$, then $P(A - B)$

$= P(A) - P(A \cap B) = P(A) - P(B)$

$[\because B \subset A \Rightarrow A \cap B = B]$

So, Statement I is correct.

P (A alone or B alone)
$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$
$= P(A) + P(B) - 2\,P(A \cap B)$
So, Statement II is wrong.
If A and B are mutually exclusive,
then $P(A \cap B) = 0$
$\Rightarrow \quad P(A \cup B) = P(A) + P(B)$
So, Statement III is correct.
Here, Statement I and III are correct.

119. A committee of three has to be chosen from a group of 4 men and 5 women. If the selection is made at random, what is the probability that exactly two members are men?

(a) $\frac{5}{14}$ (b) $\frac{1}{21}$
(c) $\frac{3}{14}$ (d) $\frac{8}{21}$

(a) Total number of selecting three members $= {}^9C_3$

Favourable numbers of selecting two members as men
$= {}^4C_2 \times {}^5C_1$

So, required probability $= \frac{{}^4C_2 \times {}^5C_1}{{}^9C_3}$

$$= \frac{\frac{4 \times 3}{2 \times 1} \times \frac{5}{1}}{\frac{9 \times 8 \times 7}{3 \times 2 \times 1}}$$

$$= \frac{2 \times 3 \times 5}{3 \times 4 \times 7} = \frac{5}{14}$$

120. The standard deviation σ of the first N natural numbers can be obtained using which one of the following formulae?

(a) $\sigma = \frac{N^2 - 1}{12}$
(b) $\sigma = \sqrt{\frac{N^2 - 1}{12}}$
(c) $\sigma = \sqrt{\frac{N - 1}{12}}$
(d) $\sigma = \sqrt{\frac{N^2 - 1}{6N}}$

(b) $\because \sigma^2 = \frac{1}{N} \Sigma X_i^2 - (\bar{X})^2$

$$= \frac{1}{N}(1^2 + 2^2 + \ldots + N^2) - \left[\frac{1}{N}(1 + 2 + 3 + \ldots + N)\right]^2$$

$$= \frac{1}{N} \times \frac{N(N+1)(2N+1)}{6} - \left[\frac{(N+1)}{2}\right]^2$$

$$= \frac{N^2 - 1}{12}$$

$$\sigma = \sqrt{\frac{N^2 - 1}{12}}$$