Application of Derivatives

Multiple Choice Questions

Choose and write the correct option in the following questions.

1. The interval in which the functions f given by $f(x) = x^2 e^{-x}$ is strictly increasing, is [*CBSE 2020 (65/2/1)*]

   (*a*) $(-\infty, \infty)$ (*b*) $(-\infty, 0)$ (*c*) $(2, \infty)$ (*d*) $(0, 2)$

2. The intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is strictly increasing in [*CBSE Sample Paper 2022 (Term-1)*]

   (*a*) $(-\infty, 2) \cup (2, \infty)$ (*b*) $(2, \infty)$ (*c*) $(-\infty, 2)$ (*d*) $(-\infty, 2] \cup (2, \infty)$

3. $f(x) = x^x$ has a stationary point at

   (*a*) $x = e$ (*b*) $x = \frac{1}{e}$ (*c*) $x = 1$ (*d*) $x = \sqrt{e}$

4. The maximum value of $\left(\frac{1}{x}\right)^x$ is [*CBSE 2021-22 (65/2/4) (Term-1)*]

   (*a*) e (*b*) e^e (*c*) $e^{1/e}$ (*d*) $\left(\frac{1}{e}\right)^{1/e}$

5. The maximum value of $[x(x-1)+1]^{\frac{1}{3}}, 0 \le x \le 1$ is [*CBSE Sample Paper 2022 (Term-1)*]

   (*a*) 0 (*b*) $\frac{1}{2}$ (*c*) 1 (*d*) $\sqrt[3]{\frac{1}{3}}$

6. The area of a trapezium is defined by function f and given by $f(x) = (10 + x)\sqrt{100 - x^2}$, then the area when it is maximised is [*CBSE Sample Paper 2022 (Term-1)*]

   (*a*) $75\ \text{cm}^2$ (*b*) $7\sqrt{3}\ \text{cm}^2$ (*c*) $75\sqrt{3}\ \text{cm}^2$ (*d*) $5\ \text{cm}^2$

7. A ladder, 5 meter long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is [*NCERT Exemplar*]

   (*a*) $\frac{1}{10}$ radian/sec (*b*) $\frac{1}{20}$ radian/sec (*c*) 20 radian/sec (*d*) 10 radian/sec

8. The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is

   (*a*) 10π (*b*) 12π (*c*) 8π (*d*) 11π

9. The total revenue in rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is

   (*a*) 116 (*b*) 96 (*c*) 90 (*d*) 126

10. If x is real, the minimum value of $x^2 - 8x + 17$ is [*NCERT Exemplar*]

    (*a*) -1 (*b*) 0 (*c*) 1 (*d*) 2

11. The function $f(x) = 2x^3 - 15x^2 + 36x + 6$ is increasing in the interval [*CBSE 2021-22 (65/2/4) (Term-1)*]

    (*a*) $(-\infty, 2) \cup (3, \infty)$ (*b*) $(-\infty, 2)$ (*c*) $(-\infty, 2] \cup [3, \infty)$ (*d*) $[3, \infty)$

12. In a sphere of radius r, a right circular cone of height h having maximum curved surface area is inscribed. The expression for the square of curved surface of cone is [*CBSE 2021-22 (65/2/4) (Term-1)*]

    (*a*) $2\pi r^2 rh(2rh + h^2)$ (*b*) $\pi^2 hr(2rh + h^2)$ (*c*) $2\pi^2 r(2rh^2 - h^3)$ (*d*) $2\pi^2 r^2(2rh - h^2)$

13. **The interval, in which function $y = x^3 + 6x^2 + 6$ is increasing, is:** [*CBSE 2021-22 (65/1/4) (Term-1)*]
(*a*) $(-\infty, -4) \cup (0, \infty)$ (*b*) $(-\infty, 4)$ (*c*) $(-4, 0)$ (*d*) $(-\infty, 0) \cup (4, \infty)$

14. **The value of x for which $(x - x^2)$ is maximum, is:** [*CBSE 2021-22 (65/1/4) (Term-1)*]
(*a*) $\frac{3}{4}$ (*b*) $\frac{1}{2}$ (*c*) $\frac{1}{3}$ (*d*) $\frac{1}{4}$

15. **A wire of length 20 cm is bent in the form of a sector of a circle. The maximum area that can be enclosed by the wire is:** [*CBSE 2021-22 (65/1/4) (Term-1)*]
(*a*) 20 sq cm (*b*) 25 sq cm (*c*) 10 sq cm (*d*) 30 sq cm

16. **The least value of the function $f(x) = ax + \frac{b}{x} (a > 0, b > 0, x > 0)$ is**
(*a*) $\frac{a}{b}$ (*b*) $2\sqrt{ab}$ (*c*) 0 (*d*) none of these

17. **The real function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is** [*CBSE Sample Paper 2022 (Term-1)*]
(*a*) Strictly increasing in $(-\infty, -2)$ and strictly decreasing in $(-2, \infty)$
(*b*) Strictly decreasing in $(-2, 3)$
(*c*) Strictly decreasing in $(-\infty, 3)$ and strictly increasing in $(3, \infty)$
(*d*) Strictly decreasing in $(-\infty, -2) \cup (3, \infty)$

18. **Let $f(x) = 2 \sin^3 x - 3 \sin^2 x + 12 \sin x + 5$, $0 \le x \le \frac{\pi}{2}$. Then $f(x)$ is**
(*a*) decreasing in $\left[0, \frac{\pi}{2}\right]$
(*b*) increasing in $\left[0, \frac{\pi}{2}\right]$
(*c*) increasing in $\left[0, \frac{\pi}{4}\right]$ and decreasing in $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
(*d*) none of these

19. **The value of b for which the function $f(x) = x + \cos x + b$ is strictly decreasing over R is** [*CBSE Sample Paper 2022 (Term-1)*]
(*a*) $b < 1$ (*b*) No value of b exists (*c*) $b \le 1$ (*d*) $b \ge 1$

20. **The global minimum value of $f(x) = x^4 - x^2 - 2x + 6$ is**
(*a*) 6 (*b*) 8 (*c*) 4 (*d*) does not exist

21. **The interval in which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is** [*CBSE 2023 (65/5/1)*]
(*a*) $(-1, \infty)$ (*b*) $(-2, -1)$ (*c*) $(-\infty, -2)$ (*d*) $[-1, 1]$

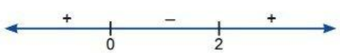
22. **If $f(x) = a(x - \cos x)$ is strictly decreasing in $\mathbb{R}$, then 'a' belongs to** [*CBSE 2023 (65/1/1)*]
(*a*) $\{0\}$ (*b*) $(0, \infty)$ (*c*) $(-\infty, 0)$ (*d*) $(-\infty, \infty)$

Answers

1. (*d*)	**2.** (*b*)	**3.** (*b*)	**4.** (*c*)	**5.** (*c*)	**6.** (*c*)	**7.** (*b*)
8. (*b*)	**9.** (*d*)	**10.** (*c*)	**11.** (*c*)	**12.** (*c*)	**13.** (*a*)	**14.** (*b*)
15. (*b*)	**16.** (*b*)	**17.** (*b*)	**18.** (*b*)	**19.** (*b*)	**20.** (*c*)	**21.** (*b*)
22. (*c*)						

Solutions of Selected Multiple Choice Questions

1. We have, $f(x) = x^2 e^{-x}$
$\Rightarrow$ $f'(x) = -x^2 e^{-x} + 2x e^{-x} = x e^{-x} (2 - x)$
for $f(x)$ to be strictly increasing, $f'(x) > 0$
$\Rightarrow$ $x e^{-x} (2 - x) > 0 \Rightarrow x(2 - x) > 0$



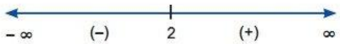
$\Rightarrow \quad x(x-2) < 0 \quad \Rightarrow \quad 0 < x < 2$

$\therefore \quad x \in (0, 2)$

$\therefore$ Option (*d*) is correct.

2. As $f(x) = x^2 - 4x + 6 \quad \Rightarrow \quad f'(x) = 2x - 4 = 2(x-2)$

Let $f'(x) = 0 \Rightarrow x = 2$



as $f'(x) > 0 \ \forall \ x \in (2, \infty)$.

$\Rightarrow$ $f(x)$ is strictly increasing in $(2, \infty)$.

$\therefore$ Option (*b*) is correct.

3. We have, $f(x) = x^x$

Let $y = x^x$

and $\log y = x\log x$

$\therefore \quad \frac{1}{y}\cdot\frac{dy}{dx} = x.\frac{1}{x} + \log x.1$ [Differentiate both sides]

$\Rightarrow \quad \frac{dy}{dx} = (1 + \log x).x^x$

$\because \quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad (1 + \log x).\ x^x = 0$

$\Rightarrow \quad \log x = -1 \quad \Rightarrow \quad \log x = \log e^{-1}$

$\Rightarrow \quad x = e^{-1} \quad \Rightarrow \quad x = \frac{1}{e}$

$\Rightarrow$ $f(x)$ has a stationary point at $x = \frac{1}{e}$.

$\therefore$ Option (*b*) is correct.

4. Let $y = \left(\frac{1}{x}\right)^x \quad \Rightarrow \quad \log y = x\ .\ \log\frac{1}{x}$

$\Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = x.\frac{1}{\frac{1}{x}}.\left(-\frac{1}{x^2}\right) + \log\frac{1}{x}.1 = -1 + \log\frac{1}{x}$ [Differentiate both sides]

$\Rightarrow \quad \frac{dy}{dx} = \left(\log\frac{1}{x} - 1\right).\left(\frac{1}{x}\right)^x$

Now, $\frac{dy}{dx} = 0 \quad \Rightarrow \log\frac{1}{x} = 1 = \log e \quad \Rightarrow \quad \frac{1}{x} = e$

$\therefore \quad x = \frac{1}{e}$

$\Rightarrow$ The maximum value of $f\left(\frac{1}{e}\right) = (e)^{1/e}$.

$\therefore$ Option (*c*) is correct.

5. Let $f(x) = [x(x-1) + 1]^{\frac{1}{3}}, \quad 0 \le x \le 1$

$f'(x) = \frac{2x - 1}{3(x^2 - x + 1)^{\frac{2}{3}}}$

Let $f'(x) = 0 \quad \Rightarrow \quad x = \frac{1}{2} \in [0, 1]$

$f(0) = 1,\ f\left(\frac{1}{2}\right) = \left(\frac{3}{4}\right)^{\frac{1}{3}}$ and $f(1) = 1$

$\Rightarrow$ Maximum value of $f(x)$ is 1.

$\therefore$ Option (*c*) is correct.

6. $f'(x) = \dfrac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}} = \dfrac{-2(x^2 + 5x - 50)}{\sqrt{100 - x^2}}$

$= \dfrac{-2(x^2 + 10x - 5x - 50)}{\sqrt{100 - x^2}} = \dfrac{-2\{x(x+10) - 5(x+10)\}}{\sqrt{100 - x^2}}$

$= \dfrac{-2(x+10)(x-5)}{\sqrt{100 - x^2}}$

$f'(x) = 0 \Rightarrow x = -10$ or 5, but $x > 0 \Rightarrow x = 5$

$f''(x) = \dfrac{2x^3 - 300x - 1000}{(100 - x)^{\frac{3}{2}}} \Rightarrow f''(5) = \dfrac{-30}{\sqrt{75}} < 0$

$\Rightarrow$ Maximum area of trapezium is $75\sqrt{3}\ \text{cm}^2$ when $x = 5$.

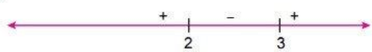
$\therefore$ Option (*c*) is correct.

11. Given $f(x) = 2x^3 - 15x^2 + 36x + 6$

$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$= 6(x^2 - 2x - 3x + 6) = 6\{x(x-2) - 3(x-2)\}$

$= 6(x-2)(x-3)$

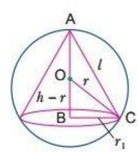


$\because \quad f'(x) > 0$ if $x \in (-\infty, 2] \cup [3, \infty)$

$\therefore$ Option (*c*) is correct.

12. Let r_1 be the radius and lateral height l of the cone of maximum curved surface area be drawn.

$\therefore$ Curved surface area of cone $= \pi r_1 l$ ...(*i*)



In ΔOBC

$r_1^2 = r^2 - (h-r)^2 = r^2 - h^2 + 2hr - r^2$

$\Rightarrow \quad r_1^2 = 2hr - h^2 \Rightarrow r_1 = \sqrt{2hr - h^2}$ ...(*ii*)

Also in ΔABC,

$l^2 = h^2 + r_1^2 = h^2 + \left(\sqrt{2hr - h^2}\right)^2 = h^2 + 2hr - h^2 = 2hr$

$\Rightarrow \quad l = \sqrt{2hr}$ ...(*iii*)

Using (*ii*) and (*iii*) in (*i*), we get

$S = \pi\sqrt{2hr - h^2}\,\sqrt{2hr}$

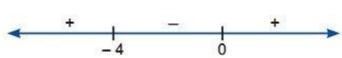
$\therefore \quad S^2 = \pi^2(2hr - h^2) \times 2hr = 2\pi^2 hr\,(2hr - h^2) = 2\pi^2 r(2h^2 r - h^3)$

$\therefore$ Option (*c*) is correct.

13. Given function, $y = x^3 + 6x^2 + 6$

$\Rightarrow \quad \dfrac{dy}{dx} = 3x^2 + 12x$

For y to be increasing, $\dfrac{dy}{dx} > 0 \quad \Rightarrow 3x^2 + 12x > 0$



$\Rightarrow \quad 3x(x+4) > 0 \quad \Rightarrow x(x+4) > 0$

$\Rightarrow \quad x < -4$ or $x > 0$

$\Rightarrow \quad x \in (-\infty, -4) \cup (0, \infty)$

$\therefore$ Option (*a*) is correct.

14. Let $y = x - x^2$

For y to be maximum or minimum.

$$\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{d(x - x^2)}{dx} = 0$$

$$\Rightarrow \quad 1 - 2x = 0 \quad \Rightarrow \quad x = \frac{1}{2}$$

Now, $\dfrac{d^2y}{dx^2} = -2 < 0$

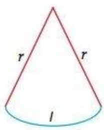
$\Rightarrow$ y will be maximum at $x = \dfrac{1}{2}$.

$\therefore$ Option (b) is correct.

15. Let r be the radius of circle and l be the arc length of the sector of the circle.

$\therefore$ Perimeter of the sector $= 2r + l$

$$20 = 2r + l \Rightarrow l = 20 - 2r$$



Now, area of sector, $A = \dfrac{1}{2} lr$

$$\Rightarrow \quad A = \frac{1}{2} \times (20 - 2r) \times r \quad \Rightarrow A = 10r - r^2$$

For area to be maximum or minimum we have

$$\frac{dA}{dr} = 0 \quad \Rightarrow 10 - 2r = 0 \quad \Rightarrow r = 5$$

Now, $\dfrac{d^2A}{dr^2} = -2 < 0$

$\Rightarrow$ Area will be maximum at $r = 5$

$\Rightarrow$ Area $= 10 \times 5 - (5)^2 = 50 - 25 = 25$ sq. cm

$\therefore$ Option (b) is correct.

16. $f(x) = ax + \dfrac{b}{x}, (a > 0, b > 0, x > 0)$

$$\Rightarrow \quad f'(x) = a - \frac{b}{x^2}$$

For maxima/minima put $f'(x) = 0$.

$$\Rightarrow \quad a - \frac{b}{x^2} = 0 \Rightarrow a = \frac{b}{x^2} \Rightarrow x^2 = \frac{b}{a} \Rightarrow x = \pm\sqrt{\frac{b}{a}}$$

$$f''(x) = \frac{2b}{x^3}$$

$$f''\left(\sqrt{\frac{b}{a}}\right) = \frac{2b}{\left(\sqrt{\frac{b}{a}}\right)^3} = \frac{2ba^{3/2}}{b^{3/2}} = \frac{2a^{3/2}}{b^{1/2}} > 0$$

$$f''\left(-\sqrt{\frac{b}{a}}\right) = \frac{b}{-\left(\sqrt{\frac{b}{a}}\right)^3} < 0$$

$x = \sqrt{\dfrac{b}{a}}$ is point of minima.

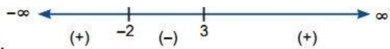
$$\therefore \quad f\left(\sqrt{\frac{b}{a}}\right) = a\left(\sqrt{\frac{b}{a}}\right) + \frac{b}{\sqrt{\frac{b}{a}}} = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$

So, $2\sqrt{ab}$ is the minimum value of f.

$\therefore$ Option (*b*) is correct.

17. $f'(x) = 6(x^2 - x - 6) = 6(x-3)(x+2)$

As $f'(x) < 0 \ \forall \ x \in (-2, 3)$



$\Rightarrow$ $f(x)$ is strictly decreasing in $(-2, 3)$.

$\therefore$ Option (*b*) is correct.

18. $f(x) = 2\sin^3 x - 3\sin^2 x + 12\sin x + 5$

$f'(x) = 6\sin^2 x\cos x - 6\sin x\cos x + 12\cos x$

$= 6\cos x\{\sin^2 x - \sin x + 2\}$

$= 6\cos x\left\{\sin^2 x - 2\sin x \times \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + 2\right\}$

$= 6\cos x\left\{\left(\sin x - \frac{1}{2}\right)^2 + \frac{7}{4}\right\} \geq 0 \ \forall \ x \in \left[0, \frac{\pi}{2}\right]$

$\therefore f(x)$ is increasing in $\left[0, \frac{\pi}{2}\right]$.

$\therefore$ Option (*b*) is correct.

19. $f'(x) = 1 - \sin x, f'(x) \geq 0 \ \forall \ x \in \mathbb{R}$

$\Rightarrow$ No value of b exists.

$\therefore$ Option (*b*) is correct.

20. $f(x) = x^4 - x^2 - 2x + 6 \quad \Rightarrow \quad f'(x) = 4x^3 - 2x - 2$

$\therefore \quad f'(x) = 0 \quad \Rightarrow \quad 4x^3 - 2x - 2 = 0$

$\Rightarrow \quad 2x^3 - x - 1 = 0 \quad \Rightarrow \quad 2x^3 - 2x^2 + 2x^2 - x - 1 = 0$

$\Rightarrow \quad 2x^3 - 2x^2 + 2x^2 - 2x + x - 1 = 0$

$\Rightarrow \quad 2x^2(x-1) + 2x(x-1) + 1(x-1) = 0$

$\Rightarrow \quad (x-1)(2x^2 + 2x + 1) = 0$

$\Rightarrow \quad x = 1$ as $2x^2 + 2x + 1 \neq 0$ for any real x.

$\therefore \quad f''(x) = 12x^2 - 2$

$\therefore \quad f''(1) = 12 - 2 = 10 > 0$

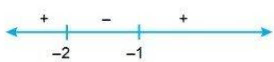
$\Rightarrow \quad x = 1$ is the point of minima.

$\therefore$ Global minimum value of $f(x)$

$= 1^4 - 1^2 - 2 \times 1 + 6 = 4.$

$\therefore$ Option (*c*) is correct.

21. $f(x) = 2x^3 + 9x^2 + 12x - 1$



$f'(x) = 6x^2 + 18x + 12 = 6(x^2 + 3x + 2)$

$= 6(x^2 + x + 2x + 2) = 6\{x(x+1) + 2(x+1)\}$

$\Rightarrow \quad f'(x) = 6(x+1)(x+2)$

$\because \quad f'(x) < 0 \ \forall \ x \in (-2, -1)$

$\therefore$ Option (*b*) is correct.

22. Given function,

$f(x) = a(x - \cos x) \quad \Rightarrow \quad f'(x) = a(1 + \sin x)$

For $f(x)$ to be decreasing

$\Rightarrow \quad f'(x) < 0 \quad \Rightarrow \quad a(1 + \sin x) < 0$

$\Rightarrow \quad a < 0 \quad (-1 \leq \sin \leq 1 \Rightarrow 0 \leq 1 + \sin x \leq 2)$

$\Rightarrow \quad a \in (-\infty, 0)$

$\therefore$ Option (*c*) is correct.

Assertion-Reason Questions

The following questions consist of two statements—Assertion(A) and Reason(R). Answer these questions selecting the appropriate option given below:

(a) Both A and R are true and R is the correct explanation for A.

(b) Both A and R are true but R is not the correct explanation for A.

(c) A is true but R is false.

(d) A is false but R is true.

1. **Assertion (A) :** The rate of change of area of a circle with respect to its radius r when $r = 6$ cm is 12π cm^2/cm.

   Reason (R) : Rate of change of area of a circle with respect to its radius r is $\frac{dA}{dr}$, where A is the area of the circle.

2. **Assertion (A) :** $f(x) = \tan x - x$ always increases.

   Reason (R) : Any function $y = f(x)$ is increasing if $\frac{dy}{dx} > 0$.

3. **Assertion (A) :** $f(x) = x^4$ is decreasing in the interval $(0, \infty)$.

   Reason (R) : Any function $y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$.

4. **Assertion (A) :** The maximum value of $\frac{\log x}{x}$ in $[2, \infty)$ is $\frac{1}{e}$.

   Reason (R) : For any function $y = f(x)$ to be increasing, $\frac{dy}{dx} > 0$.

5. **Assertion (A) :** The function $f(x) = x^2 - x$ is increasing in the interval $\left(\frac{1}{2}, \infty\right)$.

   Reason (R) : For any function $y = f(x)$ to be maximum or minimum we do $\frac{dy}{dx} = 0$.

Answers

1. (*a*) **2.** (*a*) **3.** (*d*) **4.** (*b*) **5.** (*b*)

Solutions of Assertion-Reason Questions

1. We have, $\frac{dA}{dr} = \frac{d\pi r^2}{dr} = \pi \times 2r = 2\pi r$

   $$\left.\frac{dA}{dr}\right|_{r=6} = 2\pi \times 6 = 12\pi \text{ cm}^2/\text{cm}$$

   Clearly, both Assertion(A) and Reason(R) are true and Reason(R) is the correct explanation of Assertion(A).

   $\therefore$ Option (*a*) is correct.

2. We have, $f(x) = \tan x - x$

$\Rightarrow \quad f'(x) = \sec^2 x - 1 = \tan^2 x \geq 0 \;\; \forall x \in \mathbb{R}$

$\therefore \quad f(x)$ is increasing function.

Clearly, both Assertion(A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

$\therefore$ Option (a) is correct.

3. $f(x) = x^4 \quad \Rightarrow \quad f'(x) = 4x^3$

For decreasing function

$f'(x) < 0 \quad \Rightarrow \quad 4x^3 < 0$

$\Rightarrow \quad x^3 < 0 \quad \Rightarrow \quad x < 0$

$\Rightarrow \quad x \in (-\infty, 0).$

Clearly, Assertion (A) is false and Reason (R) is true.

$\therefore$ Option (d) is correct.

4. $y = \dfrac{\log x}{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x^2} - \dfrac{\log x}{x^2} = \dfrac{1 - \log x}{x^2}$

For maxima and minima of y, $\dfrac{dy}{dx} = 0$.

$\Rightarrow \quad \dfrac{1 - \log x}{x^2} = 0 \quad \Rightarrow \quad 1 - \log x = 0$

$\Rightarrow \quad \log x = 1 = \log e$

$\Rightarrow \quad x = e,$ stationary point.

$$\frac{d^2y}{dx^2} = \frac{x^2\left(0 - \frac{1}{x}\right) - (1 - \log x) \times 2x}{x^4} = \frac{-x - 2x(1 - \log x)}{x^4}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{-e - 2e(1 - \log e)}{e^4} = -\frac{e}{e^4} = -\frac{1}{e^3} < 0$$

$\Rightarrow x = e$ is the point of maxima and maximum value is $= \dfrac{\log e}{e} = \dfrac{1}{e}$

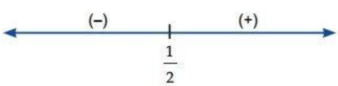
So (A) is correct statement.

Also statement R is correct but does not give the correct explanation of statement A.

$\therefore$ Option (b) is correct.

5. $\because \quad f(x) = x^2 - x$

$f'(x) = 2x - 1$



$\because \; f'(x) = 0 \quad \Rightarrow \quad x = \dfrac{1}{2}$

$\Rightarrow \quad x = \dfrac{1}{2}$ is the critical point.

$\because \quad f'(x) > 0 \;\forall\; x \in \left(\dfrac{1}{2}, \infty\right)$

$\Rightarrow \quad f(x)$ is increasing in $\left(\dfrac{1}{2}, \infty\right)$

So statement A is correct. Also, statement R is correct but does not give correct explanation of statement A.

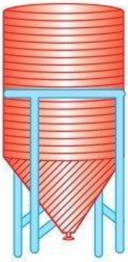
$\therefore$ Option (b) is correct.

Case-based/Data-based Questions

Each of the following questions are of 4 marks.

1. **Read the following passage and answer the following questions.** [*CBSE 2023 (65/1/1)*]

A tank, as shown in the figure below, formed using a combination of a cylinder and a cone, offers better drainage as compared to a flat bottomed tank.



A tap is connected to such a tank whose conical part is full of water. Water is dripping out from a tap at the bottom at the uniform rate of 2 cm^3/s. The semi-vertical angle of the conical tank is 45°.

(*i*) **Find the volume of water in the tank in terms of its radius** r.

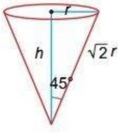
(*ii*) **Find rate of change of radius at an instant when** $r = 2\sqrt{2}$ cm.

(*iii*) (*a*) **Find the rate at which the wet surface of the conical tank is decreasing at an instant when radius** $r = 2\sqrt{2}$ cm.

OR

(*iii*) (*b*) **Find the rate of change of height 'h' at an instant when slant height is 4 cm.**

Sol. (*i*) We have,



$$\tan 45° = \frac{r}{h} \quad \Rightarrow 1 = \frac{r}{h}$$

$$h = r \quad (i) \quad \Rightarrow l = \sqrt{2}r$$

$\therefore$ Volume of cone, $V = \frac{1}{3}\pi r^2 . h = \frac{1}{3}\pi r^2 \times r$ (from (*i*))

$$= \frac{1}{3}\pi r^3$$

(*ii*) We have,

$$\frac{dV}{dt} = 2 \text{ cm}^3/\text{sec}$$

$$\Rightarrow \frac{d\left(\frac{1}{3}\pi r^3\right)}{dt} = 2 \quad \Rightarrow \quad \frac{1}{3}\pi \times 3r^2 \frac{dr}{dt} = 2$$

$$\Rightarrow \pi r^2 \frac{dr}{dt} = 2 \quad \Rightarrow \quad \frac{dr}{dt} = \frac{2}{\pi r^2}$$

$$\Rightarrow \frac{dr}{dt} = \frac{2}{\pi \times (2\sqrt{2})^2}, \text{ when } r = 2\sqrt{2}$$

$$\Rightarrow \frac{dr}{dt} = \frac{2}{8\pi} = \frac{1}{4\pi} \text{ cm/sec.}$$

(*iii*) (*a*) We have, $S = \pi r l = \pi r \times \sqrt{2} r = \sqrt{2}\pi r^2$

$\therefore$ Rate of change of surface area,

$$\frac{dS}{dt} = \frac{d\sqrt{2}\pi r^2}{dt}$$
$$= \sqrt{2}\pi \times 2r\frac{dr}{dt} = 2\sqrt{2}\pi r \times \frac{1}{4\pi}$$
$$= \frac{2\sqrt{2}\pi \times 2\sqrt{2}}{4\pi} = \frac{8\pi}{4\pi}, \text{ when } r = 2\sqrt{2} \text{ cm}$$
$$= 2 \text{ cm}^2/\text{sec.}$$

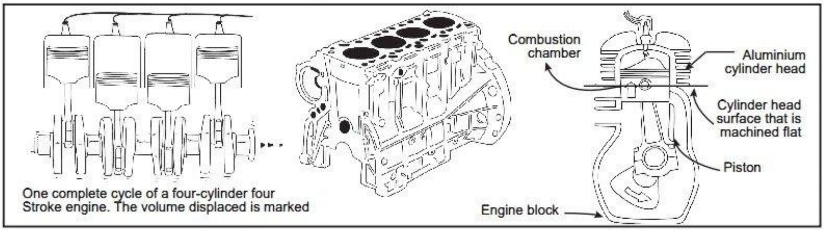
OR

(*iii*) (*b*) We have, volume of cone $= \frac{1}{3}\pi r^2.h = \frac{1}{3}\pi h^3$ $(\because h = r)$

$$V = \frac{1}{3}\pi h^3$$
$$\frac{dV}{dt} = \frac{1}{3}\pi \times 3h^2\frac{dh}{dt} \Rightarrow 2 = \pi h^2\frac{dh}{dt}$$
$$\Rightarrow \frac{dh}{dt} = \frac{2}{\pi h^2}, \text{ (Given } l = 4 \Rightarrow \sqrt{2}r = \sqrt{2}h = 4 \Rightarrow h = \frac{4}{\sqrt{2}} \Rightarrow h^2 = \frac{16}{2} = 8)$$
$$\Rightarrow \frac{dh}{dt} = \frac{2}{\pi \times \frac{16}{2}} = \frac{1}{4\pi} \text{ cm/sec.}$$

2. Read the following passage and answer the following questions. **[*CBSE 2023 (65/2/1)*]**

Engine displacement is the measure of the cylinder volume swept by all the pistons of a piston engine. The piston moves inside the cylinder bore.



The cylinder bore in the form of circular cylinder open at the top is to be made from a metal sheet of area 75π cm^2.

(*i*) If the radius of cylinder is *r* cm and height is *h* cm, then write the volume *V* of cylinder in terms of radius *r*.

(*ii*) Find $\frac{dV}{dr}$.

(*iii*) (*a*) Find the radius of cylinder when its volume is maximum.

OR

(*iii*) (*b*) For maximum volume, $h > r$. State true or false and justify.

Sol. (*i*) We have,

$$2\pi rh + \pi r^2 = 75\pi$$
$$\Rightarrow 2rh + r^2 = 75 \quad \Rightarrow 2rh = 75 - r^2$$

$$\Rightarrow \qquad h = \frac{75 - r^2}{2r}$$

$\therefore$ Volume of cylinder $V = \pi r^2 . h = \pi r^2 \times \frac{(75 - r^2)}{2r}$

$$= \frac{\pi r (75 - r^2)}{2} = \frac{\pi}{2} \times (75r - r^3)$$

(*ii*) $\frac{dV}{dr} = \frac{\pi}{2} \frac{d}{dr}(75r - r^3) = \frac{\pi}{2} \times (75 - 3r^2) = \frac{3\pi}{2}(25 - r^2)$

(*iii*) (*a*) For volume to be maximum, $\frac{dV}{dr} = 0$.

$$\Rightarrow \qquad \frac{3\pi}{2}(25 - r^2) = 0 \qquad \Rightarrow 25 - r^2 = 0$$

$$\Rightarrow r^2 = 25 \quad \Rightarrow \quad r = 5 \text{ cm}$$

$$\frac{d^2V}{dr^2} = -3\pi r$$

$$\therefore \qquad \left.\frac{d^2V}{dr^2}\right|_{r=5} = -3\pi \times 5 = -15\pi < 0$$

$\therefore$ Volume is maximum when $r = 5$ cm

OR

(*iii*) (*b*) Volume V is maximum when $r = 5$ cm

$$h = \frac{75 - r^2}{2r} = \frac{75 - (5)^2}{2 \times 5} = \frac{75 - 25}{10} = \frac{50}{10} = 5$$

$\Rightarrow \qquad h = 5$ cm

Here, $\quad h = r$

$\therefore \qquad h > r$ is false.

3. Read the following passage and answer the following questions. *[CBSE 2023 (65/2/1)]*

The use of electric vehicles will curb air pollution in the long run.

The use of electric vehicles is increasing every year and estimated electric vehicles in use at any time t is given by the function V.

$$V(t) = \frac{1}{5}t^3 - \frac{5}{2}t^2 + 25t - 2$$

Where t represents the time and $t = 1, 2, 3$ .... corresponds to year 2001, 2002, 2003, ... respectively.

(*i*) Can the above function be used to estimate number of vehicles in the year 2000? Justify.

(*ii*) Prove that the function $V(t)$ is an increasing function.

Sol. (*i*) Given, estimated electric vehicles in use at any time t is given by

$V(t) = \frac{1}{5}t^3 - \frac{5}{2}t^2 + 25t - 2$, where t represents time and $t = 1, 2, 3, \ldots$ corresponds to the year 2001, 2002, 2003, ... respectively.

$\therefore$ $t = 0$ corresponds to the year 2000

$$\therefore \quad V(0) = \frac{1}{5} \times (0)^3 - \frac{5}{2} \times (0)^2 + 25 \times 0 - 2 = -2$$

Here, estimated electric vehicles in use in the year 2000 is – 2.

So, it is not possible to calculate the estimated electric vehicles in use in the year 2000.

(*ii*) We have,

$$V(t) = \frac{1}{5}t^3 - \frac{5}{2}t^2 + 25t - 2$$

$$\Rightarrow \quad V'(t) = \frac{3}{5}t^2 - 5t + 25 = \frac{1}{5}(3t^2 - 25t + 125)$$

$$\Rightarrow \quad V'(t) = \frac{1}{5}(3t^2 - 25t + 125)$$

Now, discriminant of quadratic expression

$3t^2 - 25t + 125$ is given by

$$D = (-25)^2 - 4 \times 3 \times 125$$

$$\Rightarrow \quad D = 625 - 1500 = -875 \qquad \Rightarrow D = -875 < 0$$

We know that the quadratic expression

$p(x) = ax^2 + bx + c$ is +ve if $a > 0$ and $D = b^2 - 4ac < 0$

$$\therefore \quad 3t^2 - 25t + 125 > 0 \quad (\text{As } 3 > 0)$$

$$\Rightarrow \quad \frac{1}{5}(3t^2 - 25t + 125) > 0 \ (\text{for all } t \in R)$$

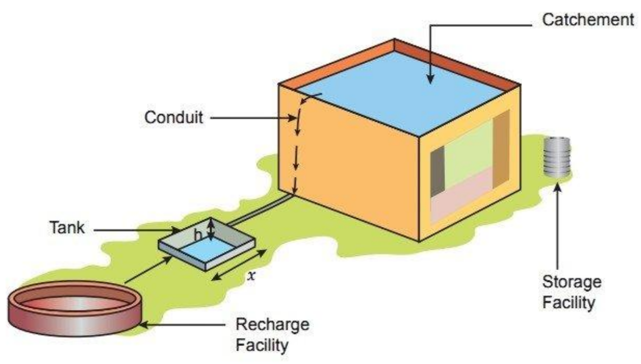
$$\Rightarrow \quad V'(t) > 0 \quad (\text{for all } t \in R)$$

$\therefore$ $V(t)$ is an increasing function. (Proved)

4. Read the following passage and answer the following questions. ***[CBSE 2023 (65/3/2)]***

In order to set up a rain water harvesting system, a tank to collect rain water is to be dug. The tank should have a square base and a capacity of 250 m^3. The cost of land is ₹ 5,000 per square metre and cost of digging increases with depth and for the whole tank, it is ₹ 40,000 h^2, where h is the depth of the tank in metres. x is the side of the square base of the tank in metres.

ELEMENTS OF A TYPICAL RAIN WATER HARVESTING SYSTEM



(*i*) **Find the total cost C of digging the tank in terms of x.**

(*ii*) **Find $\dfrac{dC}{dx}$.**

(*iii*) (*a*) **Find the value of x for which cost C is minimum**

OR

(*iii*) (*b*) **Check whether the cost function $C(x)$ expressed in terms of x is increasing or not, where $x > 0$.**

Sol. (*i*) We have, volume of the tank = 250 m^3

$$\Rightarrow \quad x^2 \times h = 250 \text{ m}^3$$

$$\Rightarrow \quad h = \frac{250}{x^2}$$

Now, total cost of digging tank = $40{,}000\,h^2 + 5000x^2$

$$\Rightarrow \quad C(x) = 40000 \times \left(\frac{250}{x^2}\right)^2 + 5000x^2$$

$$\Rightarrow \quad C(x) = 40000 \times \frac{62500}{x^4} + 5000x^2$$

(*ii*) $$\frac{dC(x)}{dx} = 40{,}000 \times 62500 \times \left(\frac{-4}{x^5}\right) + 10000x$$

$$= \frac{-4 \times 40{,}000 \times 62500}{x^5} + 10{,}000x$$

(*iii*) (*a*) For maximum/minimum cost $C(x)$

$$\frac{dC(x)}{dx} = 0 \quad \Rightarrow \quad \frac{-4 \times 40{,}000 \times 62500}{x^5} + 10{,}000x = 0$$

$$\Rightarrow \quad 10{,}000\,x = \frac{4 \times 40{,}000 \times 62500}{x^5} \quad \Rightarrow \quad x^6 = 10{,}00{,}000$$

$$\Rightarrow \quad x^6 = (10)^6 \quad \Rightarrow \quad x = 10 \text{ m}$$

Now, $$\frac{d^2}{dx^2}C(x) = 10{,}000 + \frac{20 \times 40{,}000 \times 62500}{x^6}$$

$$\Rightarrow \quad \left.\frac{d^2C(x)}{dx^2}\right|_{\text{at } x=10} > 0$$

$\therefore$ $C(x)$ is minimum at $x = 10$ m.

OR

(*iii*) (*b*) $$C(x) = 5000\,x^2 + \frac{2{,}50{,}00{,}00{,}000}{x^4}$$

$$C'(x) = 10{,}000\,x - \frac{(10)^{10}}{x^5}$$

For increasing $C'(x) > 0 \quad \Rightarrow 10{,}000\,x - \dfrac{(10)^{10}}{x^5} > 0$

$$\Rightarrow \quad 10{,}000\,x > \frac{(10)^{10}}{x^5}$$

$$\Rightarrow \quad x^6 > 10^6 \quad \Rightarrow \quad (x^3)^2 - (10^3)^2 > 0$$

$$\Rightarrow \quad (x^3 - 10^3)(x^3 + 10^3) > 0$$

$\Rightarrow \quad (x^3 - 10^3) > 0 \qquad$ (As $x > 0 \Rightarrow x^3 + 10^3 > 0$)

$\Rightarrow \quad (x - 10)(x^2 + 10x + 100) > 0$

$\Rightarrow \quad x - 10 > 0 \qquad$ (As $x > 0$ a real number)

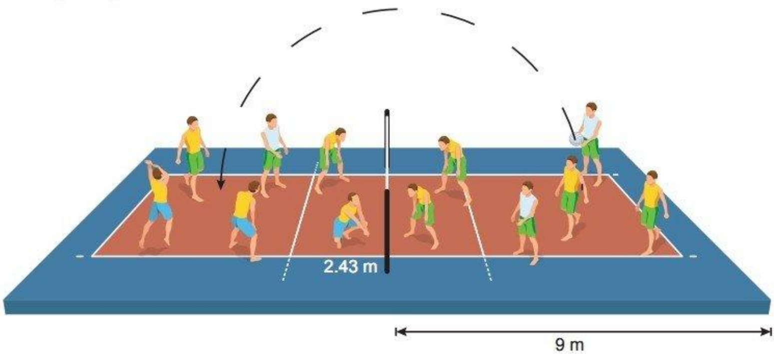
$\Rightarrow \quad x > 10$

So, for $C(x)$ to be increasing $x > 10$.

$\therefore$ $C(x)$ is not increasing for $x > 0$.

5. Read the following passage and answer the following questions. ***[CBSE 2023 (65/3/2)]***

A volleyball player serves the ball which takes a parabolic path given by the equation $h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$, where $h(t)$ is the height of ball at any time t (in seconds), $(t \geq 0)$.



(*i*) Is $h(t)$ a continuous functions ? Justify.

(*ii*) Find the time at which the height of the ball is maximum.

Sol. (*i*) Given,

$$h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1, t \geq 0$$

Here $-\frac{7}{2}t^2 + \frac{13}{2}t + 1$ is a polynomial in t, where t is time (in seconds).

$\therefore$ It is a continuous function.

Since every polynomial function is continuous function.

(*ii*) $h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$

For height to be maximum or minimum

$\frac{dh(t)}{dt} = 0 \Rightarrow \quad \frac{-7}{2} \times 2t + \frac{13}{2} = 0 \quad \Rightarrow -7t + \frac{13}{2} = 0$

$\Rightarrow \quad -14t + 13 = 0 \quad \Rightarrow 14t = 13$

$\Rightarrow \quad t = \frac{13}{14}$

$\Rightarrow \quad \frac{d^2}{dt^2}(h(t)) = -7 < 0$ at $t = \frac{13}{14}$

$\therefore \quad h(t)$ is maximum at $t = \frac{13}{14}$ seconds.

CONCEPTUAL QUESTIONS

1. **Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing.**

[*CBSE 2020 (65/3/1)*]

Sol. $f'(x) = -4 - 2x$ ½

$\Rightarrow$ $f(x)$ is increasing on $(-\infty, -2)$ ½

[*CBSE Marking Scheme 2020 (65/3/1)*]

2. **An edge of a variable cube is increasing at the rate of 5 cm per second. How fast is the volume increasing when the side is 15 cm?**

Sol. Let x be the edge of the cube and V be the volume of the cube at any time t.

Given, $\frac{dx}{dt} = 5 \text{ cm/s}, \ x = 15 \text{ cm}$

Since we know the volume of cube = $(\text{side})^3$ *i.e.*, $V = x^3$.

$$\Rightarrow \quad \frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = 3 \cdot (15)^2 \times 5 = 3375 \text{ cm}^3/\text{sec}$$

3. **If the radius of circle increasing at the rate of 0.5 cm/s, then find the rate of increase of circumference.**

Sol. Given, $\frac{dr}{dt} = 0.5 \text{ cm/s}$

$$\therefore \quad \frac{dC}{dt} = \frac{d2\pi r}{dt} = 2\pi \frac{dr}{dt}$$

$$= 2\pi \times 0.5 \text{ cm/s}$$

$$= \pi \text{ cm/sec.}$$

4. **Find the values of a for which the function $f(x) = \sin x - ax + b$ increases on $\mathbb{R}$.**

Sol. $f(x) = \sin x - ax + b$

$f'(x) = \cos x - a > 0$

if $a < \cos x$

i.e., $a \in (-\infty, -1)$

5. **Find the rate of change of volume of sphere with respect to its surface area, when radius is 2 cm.**

Sol. Let r be the radius of sphere, V be the volume and S be the surface area of sphere.

$$\because \quad V = \frac{4}{3}\pi r^3 \quad \text{and} \quad S = 4\pi r^2$$

$$\because \quad \frac{dV}{dr} = 4\pi r^2 \quad \text{and} \quad \frac{dS}{dr} = 8\pi r$$

$$\therefore \quad \frac{dV}{dS} = \frac{4\pi r^2}{8\pi r} = \frac{1}{2}r$$

$$\therefore \quad \left.\frac{dV}{dS}\right|_{r=2\text{ cm}} = \frac{2}{2} = 1 \text{ cm}^3/1\text{cm}^2$$

Very Short Answer Questions

1. **Find the values of x for which the function $f(x) = 2 + 3x - x^3$ is decreasing.** [*CBSE 2020 (65/1/2)*]

Sol.

$f(x) = 2 + 3x - x^3$

$f'(x) = 3 - 3x^2$

$= 3(1 - x^2)$

$= 3(1-x)(1+x)$

For decreasing

$f'(x) < 0$

$f'(x)$: $-\infty$ $(-)$ -1 $(+)$ 1 $(-)$ ∞

$f'(x) < 0$ when $x \in (-\infty, -1] \cup [1, \infty)$

Also $f(x)$ is continuous at $x = -1, 1$.

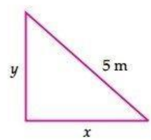
$\therefore$ The function is decreasing in $(-\infty, -1] \cup [1, \infty)$

[*Topper's Answer 2020*]

2. **A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?** [*NCERT*][*CBSE (AI) 2012*]

Sol. Let x, y be the distance of the bottom and top of the ladder respectively from the edge of the wall.

Here, $\frac{dx}{dt} = 2 \text{ cm/s}$



From figure, $x^2 + y^2 = 25$

When $x = 4$ m,

$(4)^2 + y^2 = 25$

$\Rightarrow \quad y^2 = 25 - 16 = 9 \quad \Rightarrow \quad y = 3 \text{ m}$

Now, $x^2 + y^2 = 25$

Differentiating with respect to t, we have

$$\Rightarrow \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad 4 \times 2 + 3 \times \frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{8}{3}$$

Hence, the rate of decrease of its height $= \frac{8}{3} \text{cm/s}$

3. **Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?**

[*NCERT*] [*CBSE Delhi 2011*]

Sol. Let r be the radius and h be the height of the cone so that $V = \frac{1}{3}\pi r^2 h.$

We have, $\frac{dV}{dt} = 12\text{cm}^3/\text{s} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{3}\pi r^2 h\right) = 12$ ...(*i*)

As $h = \frac{1}{6}r \quad \Rightarrow \quad r = 6h$

Putting in (*i*), we get

$$\frac{d}{dt}\left(\frac{1}{3}\pi(6h)^2 \times h\right) = 12 \quad \Rightarrow \quad \frac{d}{dt}(12\pi h^3) = 12$$

$$\Rightarrow \quad 12\pi \times 3h^2 \frac{dh}{dt} = 12 \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{3\pi h^2}$$

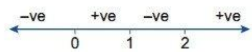
when $h = 4$ cm, $\frac{dh}{dt} = \frac{1}{3\pi(4)^2} = \frac{1}{48\pi}$ cm/s

4. Find the values of x for which $y = [x(x-2)]^2$ is an increasing function. [*NCERT*][*CBSE (AI) 2014*]

Sol. Given, $y = [x(x-2)]^2$

$$\therefore \quad \frac{dy}{dx} = 2[x(x-2)] \times (2x-2) = 4x(x-1)(x-2)$$

For increasing function, $\frac{dy}{dx} > 0$



$$4x(x-1)(x-2) > 0 \quad \Rightarrow \quad x(x-1)(x-2) > 0$$

From sign rule,

For $\frac{dy}{dx} > 0$ value of $x = 0 < x < 1$ and $x > 2$

Therefore, y is increasing $\forall\, x \in (0, 1) \cup (2, \infty)$.

5. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

[*NCERT*] [*CBSE (AI) 2011; (North) 2016*]

Sol. Given, $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$

$$\frac{dy}{dx} = \frac{(2+\cos\theta).4\cos\theta - 4\sin\theta.(0-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta - (2+\cos\theta)^2}{(2+\cos\theta)^2} = \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{+ve \times (+ve)}{+ve} \qquad \begin{bmatrix} \because \theta \in [0, \pi/2] \Rightarrow \cos\theta > 0 \\ 4 - \cos\theta \text{ is } +\text{ve as} -1 \le \cos\theta \le 1 \end{bmatrix}$$

$$\Rightarrow \quad \frac{dy}{dx} > 0$$

i.e., $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is increasing function in $\left[0, \frac{\pi}{2}\right]$.

6. A particle moves along the curve $3y = ax^3 + 1$ such that at a point with x-coordinate 1, y-coordinate is changing twice as fast at x-coordinate. Find the value of a. [*CBSE 2023 (65/2/1)*]

Sol. Given curve, $3y = ax^3 + 1$ ...(*i*)

Also, $\frac{dy}{dt} = 2\frac{dx}{dt}$ ...(*ii*)

When $x = 1$

Differentiate (*i*) w.r.t (*t*), we get

$$3\frac{dy}{dt} = a \times 3x^2 \frac{dx}{dt} \qquad \Rightarrow \frac{dy}{dt} = ax^2 \frac{dx}{dt}$$

$$\Rightarrow 2\frac{dx}{dt} = ax^2 \frac{dx}{dt}$$

$$\Rightarrow 2 = ax^2$$

$$\Rightarrow ax^2 = 2$$

$$\Rightarrow a \times (1)^2 = 2$$

$$\Rightarrow a = 2$$

7. The amount of pollution content added in air in a city due to x-diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added. *[CBSE Delhi 2013]*

Sol. We have to find $[P'(x)]_{x=3}$

Now, $P(x) = 0.005x^3 + 0.02x^2 + 30x$

$\therefore \quad P'(x) = 0.015x^2 + 0.04x + 30$

$$\Rightarrow [P'(x)]_{x=3} = 0.015 \times 9 + 0.04 \times 3 + 30$$

$$= 0.135 + 0.12 + 30$$

$$= 30.255$$

8. Show that the function $f(x) = \dfrac{16\sin x}{4 + \cos x} - x$, is strictly decreasing in $\left(\dfrac{\pi}{2}, \pi\right)$.

[CBSE 2023 (65/1/1)]

Sol. Given function,

$$f(x) = \frac{16\sin x}{4 + \cos x} - x$$

$$\Rightarrow \quad f'(x) = \frac{(4 + \cos x)\frac{d}{dx}16\sin x - 16\sin x\frac{d}{dx}(4 + \cos x)}{(4 + \cos x)^2} - 1$$

$$\Rightarrow \quad f'(x) = \frac{(4 + \cos x) \times 16\cos x - 16\sin x \times (-\sin x)}{(4 + \cos x)^2} - 1$$

$$\Rightarrow \quad f'(x) = \frac{64\cos x + 16(\cos^2 x + \sin^2 x)}{(4 + \cos x)^2} - 1$$

$$\Rightarrow \quad f'(x) = \frac{64\cos x + 16}{(4 + \cos x)^2} - 1 = \frac{64\cos x + 16 - (4 + \cos x)^2}{(4 + \cos x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{64\cos x + 16 - 16 - \cos^2 x - 8\cos x}{(4 + \cos x)^2} = \frac{56\cos x - \cos^2 x}{(4 + \cos x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{\cos x(56 - \cos x)}{(4 + \cos x)^2} < 0 \text{ for } x \in \left(\frac{\pi}{2}, \pi\right) \qquad \left\{\begin{array}{l}\text{As } \cos x < 0 \text{ in } x \in \left(\frac{\pi}{2}, \pi\right) \\ (56 - \cos x) > 0 \text{ for all } x \\ \text{and } (4 + \cos x)^2 > 0 \text{ for all } x\end{array}\right\}$$

Hence, $f(x)$ is decreasing in $\left(\dfrac{\pi}{2}, \pi\right)$.

9. **Find whether the function $f(x)=\cos\left(2x+\frac{\pi}{4}\right)$; is increasing or decreasing in the interval $\left(\frac{3\pi}{8},\frac{7\pi}{8}\right)$.** [CBSE 2019 (65/5/3)]

Sol. $f(x)=\cos\left(2x+\frac{\pi}{4}\right) \Rightarrow f'(x)=-2\sin\left(2x+\frac{\pi}{4}\right)$ ½

As $\frac{3\pi}{8}<x<\frac{5\pi}{8} \Rightarrow \frac{3\pi}{4}<2x<\frac{5\pi}{4}$

$\Rightarrow \pi<2x+\frac{\pi}{4}<\frac{3\pi}{2}$ ½

$\Rightarrow \sin\left(2x+\frac{\pi}{4}\right)<0 \Rightarrow f'(x)>0$

$\therefore$ $f(x)$ is increasing in $\left(\frac{3\pi}{8},\frac{5\pi}{8}\right)$. 1

[CBSE Marking Scheme 2019 (65/5/3)]

10. **The volume of a cube is increasing at the rate of 9 cm^3/s. How fast is its surface area increasing when the length of an edge is 10 cm?** [CBSE (AI) 2017]

Sol. Let V and S be the volume and surface area of a cube of side x cm respectively.

Given $\frac{dV}{dt}=9\ \text{cm}^3/\text{sec}$

We require $\left.\frac{dS}{dt}\right]_{x=10\text{ cm}}$

Now $V=x^3$

$\Rightarrow \frac{dV}{dt}=3x^2.\frac{dx}{dt} \Rightarrow 9=3x^2.\frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt}=\frac{9}{3x^2}=\frac{3}{x^2}$

Again, $\because S=6x^2$ [By formula for surface area of a cube]

$\Rightarrow \frac{dS}{dt}=12.x.\frac{dx}{dt}=12x.\frac{3}{x^2}=\frac{36}{x}$

$\Rightarrow \left.\frac{dS}{dt}\right]_{x=10\text{ cm}}=\frac{36}{10}=3.6\ \text{cm}^2/\text{sec}.$

11. **If x and y are the sides of two squares such that $y=x-x^2$, then find the rate of change of the area of second square with respect to the area of first square.** [NCERT Exemplar]

Sol. Since, x and y are the sides of two squares such that $y=x-x^2$.

$\therefore$ Area of the first square $(A_1)=x^2$

and area of the second square $(A_2)=y^2=(x-x^2)^2$

$\therefore \frac{dA_2}{dt}=\frac{d}{dt}(x-x^2)^2=2(x-x^2)\left(\frac{dx}{dt}-2x.\frac{dx}{dt}\right)=\frac{dx}{dt}(1-2x)2(x-x^2)$

and $\frac{dA_1}{dt}=\frac{d}{dx}x^2=2x.\frac{dx}{dt}$

$\therefore \frac{dA_2}{dA_1}=\frac{dA_2/dt}{dA_1/dt}=\frac{\frac{dx}{dt}.(1-2x)(2x-2x^2)}{2x.\frac{dx}{dt}}$

$=\frac{(1-2x)\,2x(1-x)}{2x}=(1-2x)(1-x)=1-x-2x+2x^2=2x^2-3x+1$

12. **Show that the function f defined by $f(x) = (x-1)e^x + 1$ is an increasing function for all $x > 0$.** [CBSE 2020 (65/4/1)]

Sol. $f(x) = (x-1)e^x + 1$

$\Rightarrow f'(x) = xe^x$ 1

Now $x > 0$ and $e^x > 0$ for all x

$\therefore f'(x) > 0 \Rightarrow f$ is increasing function. 1

[CBSE Marking Scheme 2020 (65/4/1)]

13. **Show that the function $f(x) = \frac{x}{3} + \frac{3}{x}$ decreases in the intervals $(-3, 0) \cup (0, 3)$.** [CBSE 2020 (65/4/3)]

Sol. $f'(x) = \frac{1}{3} - \frac{3}{x^2}$ ½

For decreasing,

$f'(x) < 0 \Rightarrow \frac{1}{3} - \frac{3}{x^2} < 0$ ½

$\Rightarrow x^2 < 9 \Rightarrow -3 < x < 3$

Since $f(x)$ is not defined at $x = 0$.

So $f(x)$ decreasing in $(-3, 0) \cup (0, 3)$. 1

[CBSE Marking Scheme 2020 (65/4/3)]

14. **Consider the statement "There exists at least one value of $b \in R$ for which $f(x) = \frac{b}{x}, b \neq 0$ is strictly increasing in $\mathbb{R} - \{0\}$." State True or False. Justify.** [CBSE 2023 (65/3/2)]

Sol. Given, $f(x) = \frac{b}{x}$, $b \neq 0$

$\Rightarrow f'(x) = \frac{-b}{x^2}, x \neq 0$ as $x \in R - \{0\}$

for increasing $f'(x) > 0$

$$\frac{-b}{x^2} > 0 \Rightarrow -b > 0$$

$$\Rightarrow b < 0 \Rightarrow b \in (-\infty, 0)$$

$\therefore$ Statement is true.

Short Answer Questions

1. **Find the intervals in which the function f given by**

$$f(x) = \tan x - 4x, \quad x \in \left(0, \frac{\pi}{2}\right) \text{ is}$$

(a) strictly increasing (b) strictly decreasing [CBSE Sample Paper 2021]

Sol. $f(x) = \tan x - 4x \Rightarrow f'(x) = \sec^2 x - 4$

(a) For $f(x)$ to be strictly increasing

$$f'(x) > 0$$

$$\Rightarrow \sec^2 x - 4 > 0 \Rightarrow \sec^2 x > 4$$

$$\Rightarrow \quad \cos^2 x < \frac{1}{4} \qquad \Rightarrow \quad \cos^2 x < \left(\frac{1}{2}\right)^2 \ |\cos x| < \frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{2} < \cos x < \frac{1}{2} \quad \Rightarrow \quad \frac{\pi}{3} < x < \frac{\pi}{2} \qquad \left[\because x \in \left(0, \frac{\pi}{2}\right)\right]$$

(*b*) For $f(x)$ to be strictly decreasing

$$f'(x) < 0$$

$$\Rightarrow \sec^2 x - 4 < 0 \qquad \Rightarrow \sec^2 x < 4$$

$$\Rightarrow \cos^2 x > \frac{1}{4} \qquad \Rightarrow \cos^2 x > \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \cos x > \frac{1}{2} \quad \left[\because x \in \left(0, \frac{\pi}{2}\right)\right] \quad \Rightarrow 0 < x < \frac{\pi}{3}$$

2. Find the intervals in which $f(x) = \sin 3x - \cos 3x, 0 < x < \pi$, is strictly increasing or strictly decreasing. **[*CBSE Delhi 2016*]**

Sol. Given function is

$$f(x) = \sin 3x - \cos 3x$$

$$f'(x) = 3\cos 3x + 3\sin 3x$$

For critical points of function $f(x)$

$$f'(x) = 0$$

$$\Rightarrow \quad 3\cos 3x + 3\sin 3x = 0 \quad \Rightarrow \quad \cos 3x + \sin 3x = 0$$

$$\Rightarrow \quad \sin 3x = -\cos 3x \quad \Rightarrow \quad \frac{\sin 3x}{\cos 3x} = -1$$

$$\Rightarrow \quad \tan 3x = -\tan\frac{\pi}{4} \quad \Rightarrow \quad \tan 3x = \tan\left(\pi - \frac{\pi}{4}\right)$$

$$\Rightarrow \quad \tan 3x = \tan\frac{3\pi}{4}$$

$$\Rightarrow \quad 3x = n\pi + \frac{3\pi}{4}, \text{ where } n = 0, \pm 1, \pm 2, \ldots.$$

Putting $n = 0, \pm 1, \pm 2, \ldots.$, we get

$$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12} \in (0, \pi)$$

Hence, required possible intervals are $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$ and $\left(\frac{11\pi}{12}, \pi\right)$

For $\left(0, \frac{\pi}{4}\right)$, $f'(x) = +\text{ve}$

For $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$, $f'(x) = -\text{ve}$

For $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$, $f'(x) = +\text{ve}$

For $\left(\frac{11\pi}{12}, \pi\right)$, $f'(x) = -\text{ve}$

Hence, given function $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$ and strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$.

3. **Find the intervals in which the function $f(x) = -3\log(1+x) + 4\log(2+x) - \frac{4}{2+x}$ is strictly increasing or strictly decreasing.** [*CBSE Sample Paper 2018*]

Sol. Given $f(x) = -3\log(1+x) + 4\log(2+x) - \frac{4}{2+x}$

$$\Rightarrow f'(x) = \frac{-3}{1+x} + \frac{4}{2+x} + \frac{4}{(2+x)^2} = \frac{-3(2+x)^2 + 4(1+x)(2+x) + 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{-3(4+4x+x^2) + 4(2+x+2x+x^2) + 4 + 4x}{(1+x)(2+x)^2}$$

$$= \frac{-12 - 12x - 3x^2 + 8 + 12x + 4x^2 + 4 + 4x}{(1+x)(2+x)^2}$$

$$f'(x) = \frac{x(x+4)}{(1+x)(2+x)^2}$$

Now, $f'(x) = 0$

$\Rightarrow \quad \frac{x(x+4)}{(1+x)(2+x)^2} = 0 \quad \Rightarrow \quad x(x+4) = 0$

$\Rightarrow \quad x = 0 \quad [\because x \neq -4$ as $f(x)$ is defined on $(-1, \infty)]$

Hence, required intervals are $(-1, 0)$ and $(0, \infty)$.

For $(-1, 0)$

$f'(x) = \frac{(-\text{ve}) \times (+\text{ve})}{(+\text{ve}) \times (+\text{ve})} = -\text{ve} \quad \Rightarrow \quad f(x)$ is strictly decreasing in $(-1, 0)$.

For $(0, \infty)$

$f'(x) = \frac{(+\text{ve}) \times (+\text{ve})}{(+\text{ve}) \times (+\text{ve})} = +\text{ve} \quad \Rightarrow \quad f(x)$ is strictly increasing in $(0, \infty)$.

i.e., $f(x)$ is strictly decreasing on $(-1, 0)$ and strictly increasing on $(0, \infty)$.

4. **Find the intervals in which the function $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$ is**
(*a*) strictly increasing (*b*) strictly decreasing. [*CBSE (F) 2014*]

Sol. Here, $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$

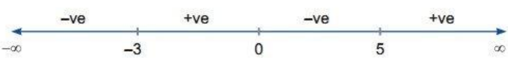
$\Rightarrow \quad f'(x) = 6x^3 - 12x^2 - 90x \quad \Rightarrow \quad f'(x) = 6x(x^2 - 2x - 15) = 6x(x+3)(x-5)$

Now for critical point $f'(x) = 0$

$6x(x+3)(x-5) = 0 \quad \Rightarrow \quad x = 0, -3, 5$

i.e., $-3, 0, 5$ are critical points which divides domain R of given function into four disjoint sub intervals $(-\infty, -3), (-3, 0), (0, 5), (5, \infty)$.

For $(-\infty, -3)$



$f'(x) = +\text{ve} \times (-\text{ve}) \times (-\text{ve}) \times (-\text{ve}) = -\text{ve}$

i.e., $f(x)$ is decreasing in $(-\infty, -3)$.

For $(-3, 0)$

$f'(x) = +\text{ve} \times (-\text{ve}) \times (+\text{ve}) \times (-\text{ve}) = +\text{ve}$

i.e., $f(x)$ is increasing in $(-3, 0)$.

For (0, 5)

$$f'(x) = +\text{ve} \times (+\text{ve}) \times (+\text{ve}) \times (-\text{ve}) = -\text{ve}$$

i.e., $f(x)$ is decreasing in (0, 5).

For $(5, \infty)$

$$f'(x) = +\text{ve} \times (+\text{ve}) \times (+\text{ve}) \times (+\text{ve}) = +\text{ve}$$

i.e., $f(x)$ is increasing in $(5, \infty)$.

Hence, $f(x)$ is (*a*) strictly increasing in $(-3, 0) \cup (5, \infty)$

(*b*) strictly decreasing in $(-\infty, -3) \cup (0, 5)$.

5. Show that $y = \log(1+x) - \frac{2x}{2+x}, x > -1$ is an increasing function of x throughout its domain. [*CBSE (F) 2012*]

Sol. Here, $f(x) = \log(1+x) - \frac{2x}{2+x}$ [where $y = f(x)$]

$$\Rightarrow \quad f'(x) = \frac{1}{1+x} - 2\left[\frac{(2+x).1 - x}{(2+x)^2}\right]$$

$$= \frac{1}{1+x} - \frac{2(2+x-x)}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$= \frac{4 + x^2 + 4x - 4 - 4x}{(x+1)(x+2)^2} = \frac{x^2}{(x+1)(x+2)^2}$$

For $f(x)$ being increasing function

$$f'(x) > 0$$

$$\Rightarrow \quad \frac{x^2}{(x+1)(x+2)^2} > 0 \quad \Rightarrow \quad \frac{1}{x+1} \cdot \frac{x^2}{(x+2)^2} > 0$$

$$\Rightarrow \quad \frac{1}{x+1} > 0 \qquad \left[\frac{x^2}{(x+2)^2} > 0\right]$$

$$\Rightarrow \quad x + 1 > 0 \quad \text{or} \quad x > -1$$

i.e., $f(x) = y = \log(1+x) - \frac{2x}{2+x}$ is increasing function in its domain $x > -1$ *i.e.*, $(-1, \infty)$.

6. Show that $f(x) = 2x + \cot^{-1} x + \log\left(\sqrt{1+x^2} - x\right)$ is increasing in R. [*NCERT Exemplar*]

Sol. We have, $f(x) = 2x + \cot^{-1}x + \log\left(\sqrt{1+x^2} - x\right)$

$$f'(x) = 2 + \left(\frac{-1}{1+x^2}\right) + \frac{1}{\left(\sqrt{1+x^2} - x\right)}\left(\frac{1}{2\sqrt{1+x^2}} \cdot 2x - 1\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{1}{\left(\sqrt{1+x^2} - x\right)} \cdot \frac{\left(x - \sqrt{1+x^2}\right)}{\sqrt{1+x^2}} = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{2 + 2x^2 - 1 - \sqrt{1+x^2}}{1+x^2} = \frac{1 + 2x^2 - \sqrt{1+x^2}}{1+x^2}$$

For increasing function, $f'(x) \geq 0$

$$\frac{1 + 2x^2 - \sqrt{1+x^2}}{1+x^2} \geq 0 \quad \Rightarrow \quad 1 + 2x^2 \geq \sqrt{1+x^2}$$

$$\Rightarrow \quad (1 + 2x^2)^2 \geq 1 + x^2 \quad \Rightarrow \quad 1 + 4x^4 + 4x^2 \geq 1 + x^2$$

$$\Rightarrow \quad 4x^4 + 3x^2 \geq 0 \quad \Rightarrow \quad x^2(4x^2 + 3) \geq 0$$

It is true for any real value of x.

Hence, $f(x)$ is increasing in R.

Long Answer Questions

1. **Find the minimum value of $(ax + by)$, where $xy = c^2$.** *[CBSE Delhi 2015, 2020 (65/5/1)]*

Sol. Let $S = ax + by$, where $y = \frac{c^2}{x}$ $\quad \therefore \quad S = ax + \frac{bc^2}{x}$ **1**

$$\frac{dS}{dx} = a - \frac{bc^2}{x^2} \Rightarrow a = \frac{bc^2}{x^2}$$ **1**

$$\Rightarrow \frac{dS}{dx} = 0 \Rightarrow x^2 = \frac{bc^2}{a} \text{ or } x = \sqrt{\frac{b}{a}}.c$$ **1½**

$$\left.\frac{d^2S}{dx^2}\right|_{x=\sqrt{\frac{b}{a}}.c} = \left.\frac{2bc^2}{x^3}\right|_{x=\sqrt{\frac{b}{a}}.c} = 2bc^2\left[\sqrt{\frac{a}{b}}\frac{1}{c}\right]^3 > 0 \text{ for } a, b, c > 0 \text{ and } x = \sqrt{\frac{b}{a}}.c$$ **1½**

$\therefore$ Minimum value $= a\sqrt{\frac{b}{a}}.c + b.\frac{c^2}{c}\sqrt{\frac{a}{b}} = 2\sqrt{ab}.c$ **1**

[CBSE Marking Scheme 2020 (65/1/1)]

Detailed Solution:

Let $z = ax + by$ ...(*i*)

Given $xy = c^2 \quad \Rightarrow \quad y = \frac{c^2}{x}$

Putting $y = \frac{c^2}{x}$ in equation (*i*), we have

$$z = ax + \frac{bc^2}{x}$$

For z to be maximum or minimum

$$\frac{dz}{dx} = a - \frac{bc^2}{x^2} = 0 \quad \Rightarrow \quad a = \frac{bc^2}{x^2}$$

$$\Rightarrow \quad x^2 = \frac{bc^2}{a} \quad \Rightarrow \quad x = \pm c\sqrt{\frac{b}{a}}$$

Now, $\frac{d^2z}{dx^2} = \frac{2bc^2}{x^3}$

$\therefore$ at $x = c\sqrt{\frac{b}{a}}$, $\frac{d^2z}{dx^2} = \frac{2bc^2}{\left(c\sqrt{\frac{b}{a}}\right)^3} > 0$

$\therefore$ z will be minimum at $x = c\sqrt{\frac{b}{a}}$

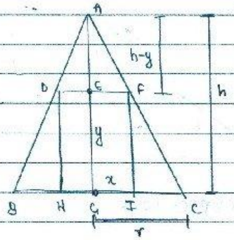
$\therefore \quad y = \frac{c^2}{x} = \frac{c^2}{c\sqrt{\frac{b}{a}}} = c\sqrt{\frac{a}{b}}$

$\therefore$ Minimum value of $z = ax + by$

$$= a \times c\sqrt{\frac{b}{a}} + b \times c\sqrt{\frac{a}{b}} = c\sqrt{ab} + c\sqrt{ab} = 2c\sqrt{ab}$$

2. **Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone.** *[CBSE 2020 (65/1/1)]*

Sol.



let the cone be represented by ABC and the cylinder have radius and height x and y respectively.

$\therefore$ As ΔAEF and ΔAGC are similar

$$\frac{h-y}{h} = \frac{x}{r}$$

$$rh - ry = xh \Rightarrow \frac{rh - xh}{r} = y.$$

Volume of the cylinder $= V = \pi x^2 y$

$$\Rightarrow V = \frac{\pi x^2}{r}(rh - xh)$$

$$V = \pi x^2 h - \frac{\pi x^3 h}{r}$$

Differentiating with respect to x

$$\Rightarrow \frac{dV}{dx} = 2\pi x h - \frac{3\pi x^2 h}{r} \quad -(1)$$

for maximum volume $\frac{dV}{dx} = 0$

$$\Rightarrow 2\pi x h = \frac{3\pi x^2 h}{r}$$

$$\Rightarrow 2r = 3x \Rightarrow x = \frac{2}{3}r$$

Differentiating equation (1) with respect to x

$$\Rightarrow \frac{d^2V}{dx^2} = 2\pi h - \frac{6\pi x h}{r}$$

$$\left(\frac{d^2V}{dx^2}\right)_{(x=\frac{2}{3}r)} = 2\pi h - \frac{6\pi 2 r h}{3r} = 2\pi h - 4\pi h = -2\pi h < 0$$

for maximum volume $x = \frac{2}{3}r$

$$y = \frac{rh - xh}{r} = \frac{rh - \frac{2}{3}rh}{r} = \frac{h}{3}$$

$\therefore$ Height of the right cylinder with maximum volume is $\frac{1}{3}$rd height of cone.

Volume of the cone $= \frac{1}{3}\pi r^2 h$

Maximum Volume of the cylinder $= \pi x^2 y = \pi \frac{4r^2}{9}\frac{h}{3} = \frac{4\pi r^2 h}{27} = \frac{4}{9}\left(\frac{1}{3}\pi r^2 h\right)$

$= \frac{4}{9}$(Volume of the cone)

Hence Proved

[*Topper's Answer 2020*]

3. **Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers.** *[CBSE 2023 (65/5/1)]*

Sol. Let x and y are the numbers.

From question

$$x + y = 5 \quad \text{...(i)}$$

Let $$x^3 + y^3 = C \text{ (say)} \quad \text{...(ii)}$$

and $$x^2 + y^2 = S \text{ (say)} \quad \text{...(iii)}$$

From (*i*), $y = 5 - x$,

Using in (*ii*), we get

$$C = x^3 + y^3 = x^3 + (5 - x)^3$$

$$\Rightarrow \quad C = x^3 + 125 - 75x + 15x^2 - x^3$$

$$\Rightarrow \quad C = 15x^2 - 75x + 125$$

$$\therefore \quad \frac{d(C)}{dx} = 30x - 75 = 15\,(2x - 5)$$

$$\therefore \quad \frac{d(C)}{dx} = 0 \quad \Rightarrow \quad 15\,(2x - 5) = 0$$

$$\Rightarrow \quad x = \frac{5}{2} \text{ is the critical point.}$$

$$\therefore \quad \frac{d^2(C)}{dx^2} = 30$$

$$\therefore \quad \left.\frac{d^2(C)}{dx^2}\right|_{x = 5/2} = 30 > 0$$

Sum of cubes is minimum when $x = \frac{5}{2}$

If $x = \frac{5}{2}$, $y = 5 - x = 5 - \frac{5}{2} = \frac{5}{2}$

$\therefore$ Sum of the squares of the numbers $S = x^2 + y^2$

$$= \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 = \frac{25}{4} + \frac{25}{4} = \frac{50}{4} = \frac{25}{2}$$

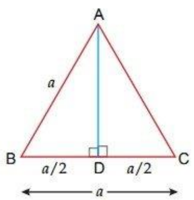
4. **The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing.** *[CBSE 2023 (65/5/1)]*

Sol. In an equilateral triangle, median = height.

Let $AB = BC = CA = a$.

Let AD be the median of ΔABC.

$\therefore BD = \frac{a}{2}$



And $AD = \sqrt{AB^2 - BD^2} = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \sqrt{a^2 - \frac{a^2}{4}}$

$$\Rightarrow \quad AD = \sqrt{\frac{3a^2}{4}} = \frac{\sqrt{3}}{2}a$$

Given that $\frac{d(AD)}{dt} = 2\sqrt{3}$ cm/s

$$\frac{da}{dt} = ?$$

$\because \qquad AD = \frac{\sqrt{3}}{2}a$

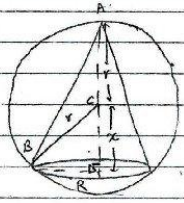
$\therefore \qquad \frac{d}{dt}(AD) = \frac{d}{dt}\left(\frac{\sqrt{3}}{2}a\right) = \frac{\sqrt{3}}{2}\frac{da}{dt}$

$\Rightarrow \qquad 2\sqrt{3} = \frac{\sqrt{3}}{2} \times \frac{da}{dt}$

$\Rightarrow \qquad \frac{da}{dt} = 2\sqrt{3} \times \frac{2}{\sqrt{3}} = 4 \text{ cm/s}$

5. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius *r* is $\frac{4r}{3}$. Also find the maximum volume of cone. [*NCERT*][*CBSE 2019 (65/1/2)*]

Sol.



Consider a sphere of radius r.

Then, height of cone = $r + x = H$

Radius of cone = $\sqrt{r^2 - x^2} = R$

Volume of cone = $V = \frac{1}{3}\pi R^2 H$

$V = \frac{1}{3}\pi (r^2 - x^2)(r + x)$

$= \frac{\pi}{3}(r^3 - rx^2 + xr^2 - x^3)$

$\frac{dV}{dx} = \frac{\pi}{3}(-2rx + r^2 - 3x^2)$

For critical pt $\frac{dV}{dx} = 0$

$r^2 - 2rx - 3x^2 = 0$

$r^2 - 3rx + rx - 3x^2 = 0$

$(r + x)(r - 3x) = 0$

x cannot be –ve $\Rightarrow \boxed{\frac{r}{3} = x}$ is critical pt

$\frac{d^2V}{dx^2} = \frac{\pi}{3}(-2r - 6x)$

$\left.\frac{d^2V}{dx^2}\right]_{x = r/3} = \frac{\pi}{3}[-2r - 2r] = \frac{-4r\pi}{3} < 0$

Using Second Derivative test, $\therefore \boxed{x = \frac{r}{3}}$ is a point of maxima

$H = r + x$

$\boxed{H = \frac{4r}{3}}$ Hence proved

Max. volume of cone $= V_{max} = \frac{\pi}{3}(r+x)(r^2-x^2)$

$$= \frac{\pi}{3}\left(\frac{4r}{3}\right)\left(r^2-\frac{r^2}{9}\right)$$

$$= \frac{4\pi r^3}{81}\cdot 8 = \frac{32\pi r^3}{81}\ (\text{unit})^3$$

$$V_{max} = \frac{32\pi r^3}{81}\ (\text{unit})^3$$

[*Topper's Answer 2019*]

6. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹70 per square metre for the base and ₹45 per square metre for the sides, what is the cost of least expensive tank?

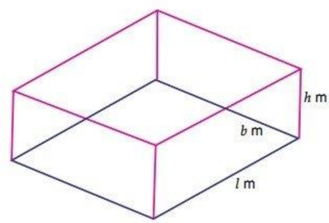
[*NCERT*] [*CBSE 2019 (65/1/1)*]

Sol. Let l, b and h metre be the length, breadth and height of the tank respectively.

Given $h = 2$ m

and volume of tank $= l \times b \times h$

$\Rightarrow \quad 8 = l \times b \times 2 \Rightarrow lb = 4 \quad \Rightarrow \quad b = \frac{4}{l}$



Now, area of the base, $\quad lb = 4\text{ m}^2$

and, area of four walls, $\quad A = 2(l+b) \times h$

$$= 2\left(l+\frac{4}{l}\right)\times 2 \Rightarrow A = 4\left(l+\frac{4}{l}\right)$$

For minimum cost

$$\frac{dA}{dl} = 0 \quad \Rightarrow 4\left(1-\frac{4}{l^2}\right) = 0$$

$\Rightarrow \quad \frac{4}{l^2} = 1 \quad \Rightarrow l^2 = 4 \quad \Rightarrow \quad l = 2\text{m}$

$\therefore \quad l = 2\text{m}$ ($\because$ length cannot be negative) and $b = \frac{4}{l} = \frac{4}{2} = 2\text{m}$

Now, $\quad \frac{d^2A}{dl^2} = \frac{32}{l^2} = \frac{32}{4} = 8 > 0$

$\therefore$ Area will be minimum when $l = 2\text{m},\ b = 2\text{m},\ h = 2\text{m}$

$\therefore$ Cost of building of tank $= 70 \times (l \times b) = 70 \times 2 \times 2 =$ ₹ 280

and cost of building the walls $= 45 \times 2h\,(l+b) = 90 \times 2\,(2+2) =$ ₹ 720

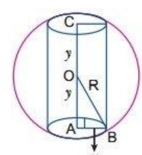
$\therefore$ Total cost for building the tank $= 280 + 720 =$ ₹ 1,000

7. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. **[*NCERT*] [*CBSE 2019 (65/2/1)*]**

Sol. Let x be radius and $(y+R)$ be the height of cylinder given radius of sphere be R.

In ΔOAB, we have,

$$OB^2 = OA^2 + AB^2$$



$\Rightarrow \quad R^2 = y^2 + x^2 \Rightarrow x^2 + y^2 = R^2 \quad \Rightarrow \quad x^2 = R^2 - y^2 \quad ...(i)$

Now, volume of cylinder $= \pi x^2 \times 2y$

$\Rightarrow \quad V = \pi(R^2 - y^2) \times 2y$

For volume to be maximum or minimum

$$\frac{dV}{dy}=0 \qquad \Rightarrow \qquad 2\pi\{(R^2-y^2)\times 1+y\times(-2y)\}=0$$

$$\Rightarrow \quad R^2-y^2-2y^2=0 \qquad \Rightarrow \qquad R^2-3y^2=0 \qquad \Rightarrow \qquad R^2=3y^2$$

$$\Rightarrow \quad y^2=\frac{R^2}{3} \quad \Rightarrow \quad y=\frac{R}{\sqrt{3}}$$

$$\because \quad \frac{dV}{dy}=2\pi(R^2-3y^2)$$

$$\therefore \quad \left.\frac{d^2V}{dy^2}\right|_{\left(\text{at } y=\frac{R}{\sqrt{3}}\right)}=2\pi(-6y)=-12\pi y=\frac{-12R\pi}{\sqrt{3}}<0$$

$\therefore$ Volume will be maximum when $y=\dfrac{R}{\sqrt{3}}$.

$\therefore$ Height of cylinder $=2y=\dfrac{2R}{\sqrt{3}}$

and maximum volume $=\pi(R^2-y^2)\times 2y$

$$=\pi\left(R^2-\frac{R^2}{3}\right)\times\frac{2R}{\sqrt{3}}=\pi\times\frac{2R^2}{3}\times\frac{2R}{\sqrt{3}}=\frac{4\pi R^3}{3\sqrt{3}}$$

8. Find the maximum and minimum values of $f(x)=\sec x+\log\cos^2 x,\ 0<x<2\pi$. **[*CBSE (South) 2016*]**

Sol. We have $f(x)=\sec x+\log\cos^2 x$

$$f'(x)=\sec x.\tan x+\frac{1}{\cos^2 x}.2\cos x(-\sin x)=\sec x.\tan x-2\tan x=\tan x(\sec x-2)$$

For critical point

$f'(x)=0$

$$\Rightarrow \tan x(\sec x-2)=0 \qquad \Rightarrow \qquad \tan x=0 \text{ or } \sec x-2=0$$

$$\Rightarrow \quad x=n\pi \text{ or } \sec x=2 \qquad \Rightarrow \qquad x=n\pi \text{ or } \cos x=\frac{1}{2}$$

$$\Rightarrow \quad x=n\pi \text{ or } \cos x=\cos\frac{\pi}{3} \qquad \Rightarrow \qquad x=n\pi \text{ or } x=2n\pi\pm\frac{\pi}{3},\ n=0,\pm1,\pm2.......$$

Thus possible value of x in interval $0<x<2\pi$ are $x=\dfrac{\pi}{3},\pi,\dfrac{5\pi}{3}$

Now, $f\left(\dfrac{\pi}{3}\right)=\sec\dfrac{\pi}{3}+\log\cos^2\dfrac{\pi}{3}=2+\log\left(\dfrac{1}{2}\right)^2$

$=2+2(\log 1-\log 2)=2-2\log 2=2(1-\log 2)$ $\qquad [\because \log 1=0]$

$f(\pi)=\sec\pi+\log\cos^2\pi=-1+\log(-1)^2=-1$

$$f\left(\frac{5\pi}{3}\right)=\sec\frac{5\pi}{3}+2\log\cos\frac{5\pi}{3}=\sec\left(2\pi-\frac{\pi}{3}\right)+2\log\cos\left(2\pi-\frac{\pi}{3}\right)$$

$$=\sec\frac{\pi}{3}+2\log\cos\frac{\pi}{3}=2+2\log\frac{1}{2}$$

$$=2+2(\log 1-\log 2)=2-2\log 2=2(1-\log 2)$$

Hence, maximum value of $f(x)=2(1-\log 2)$ and minimum value of $f(x)=-1$.

9. Find the absolute maximum and absolute minimum values of the function f given by $f(x)=\sin^2 x-\cos x,\ x\in[0,\pi]$. **[*CBSE Panchkula 2015*]**

Sol. Here, $f(x)=\sin^2 x-\cos x$

$f'(x)=2\sin x.\cos x+\sin x$

$\Rightarrow \quad f'(x)=\sin x(2\cos x+1)$

For critical point: $f'(x) = 0$

$\Rightarrow \quad \sin x(2\cos x + 1) = 0 \quad \Rightarrow \quad \sin x = 0$ or $\cos x = -\frac{1}{2}$

$\Rightarrow \quad x = 0$ or $\cos x = \cos\frac{2\pi}{3} \quad \Rightarrow \quad x = 0$ or $x = 2n\pi \pm \frac{2\pi}{3}$, where $n = 0, \pm1, \pm2 \ldots$

$\Rightarrow \quad x = 0$ or $x = \frac{2\pi}{3}$ other values does not belong to $[0, \pi]$.

For absolute maximum or minimum values:

$$f(0) = \sin^2 0 - \cos 0 = 0 - 1 = -1$$

$$f\left(\frac{2\pi}{3}\right) = \sin^2\frac{2\pi}{3} - \cos\frac{2\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

$$f(\pi) = \sin^2\pi - \cos\pi = 0 - (-1) = 1$$

Hence, absolute maximum value = $\frac{5}{4}$ and absolute minimum value = -1.

10. If the function $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1$, where $m > 0$ attains its maximum and minimum at p and q respectively such that $p^2 = q$, then find the value of m. ***[CBSE Patna 2015]***

Sol. Given, $f(x) = 2x^3 - 9mx^2 + 12m^2x + 1$

$\Rightarrow \quad f'(x) = 6x^2 - 18mx + 12m^2$

For extremum value of $f(x)$, $f'(x) = 0$

$\Rightarrow \quad 6x^2 - 18mx + 12m^2 = 0$

$\Rightarrow \quad x^2 - 3mx + 2m^2 = 0$

$\Rightarrow \quad x^2 - 2mx - mx + 2m^2 = 0 \quad \Rightarrow \quad x(x - 2m) - m(x - 2m) = 0$

$\Rightarrow \quad (x - m)(x - 2m) = 0 \quad \Rightarrow \quad x = m$ or $x = 2m$

Now, $f''(x) = 12x - 18m$

$\Rightarrow \quad f''(x)$ at $[x = m] = f''(m) = 12m - 18m = -6m < 0$

And, $f''(x)$ at $[x = 2m] = f''(2m) = 24m - 18m = 6m > 0$

Hence, $f(x)$ attains maximum and minimum value at m and $2m$ respectively.

$\Rightarrow \quad m = p$ and $2m = q$

But, $p^2 = q$ [Given]

$\therefore \quad m^2 = 2m \quad \Rightarrow \quad m^2 - 2m = 0$

$\Rightarrow \quad m(m - 2) \quad \Rightarrow \quad m = 0$ or $m = 2$

$\Rightarrow \quad m = 2$ as $m > 0$

11. The sum of the surface areas of a cuboid with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes. ***[CBSE (F) 2016]***

Sol. Let r be the radius of sphere and S, V be the sum of surface area and volume of cuboid and sphere.

Now $V = \left(x \cdot 2x \cdot \frac{x}{3}\right) + \frac{4}{3}\pi r^3$

$\Rightarrow \quad V = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V = \frac{2}{3}(x^3 + 2\pi r^3)$

$\left[\because \quad S = 2\left[x.2x + x.\frac{x}{3} + \frac{x}{3}.2x\right] + 4\pi r^2\right.$

$\Rightarrow \quad S = \frac{18x^2}{3} + 4\pi r^2 = 6x^2 + 4\pi r^2$

$\left.\Rightarrow \quad x^2 = \frac{S - 4\pi r^2}{6} \quad \Rightarrow \quad x^3 = \left(\frac{S - 4\pi r^2}{6}\right)^{3/2}\right]$

$\Rightarrow \quad V = \frac{2}{3}\left\{\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{3}{2}} + 2\pi r^3\right\}$

$\Rightarrow \quad \frac{dV}{dr} = \frac{2}{3}\left\{\frac{3}{2}\left(\frac{S - 4\pi r^2}{6}\right)^{\frac{1}{2}} \cdot \frac{1}{6} \cdot (-8\pi r) + 6\pi r^2\right\}$

For maximum or minimum value

$$\frac{dV}{dr} = 0$$

$$\Rightarrow \quad \frac{2}{3}\left\{-2\pi r\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}} + 6\pi r^2\right\} = 0 \quad \Rightarrow \quad \left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}} = \frac{6\pi r^2}{2\pi r}$$

$$\Rightarrow \quad \left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}} = 3r \quad \Rightarrow \quad r = \frac{1}{3}\cdot\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}$$

Obviously, $\left.\frac{d^2V}{dr^2}\right]_{r=\frac{1}{3}\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}} = +\text{ve}$

$\therefore$ V is minimum when $r = \frac{1}{3}\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}$

$$\Rightarrow \quad 3r = \left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}} \quad \Rightarrow \quad 9r^2 = \left(\frac{S-4\pi r^2}{6}\right) \qquad \text{[Squaring both sides]}$$

$$\Rightarrow \quad 54r^2 = S - 4\pi r^2 \quad \Rightarrow \quad 54r^2 = 6x^2 + 4\pi r^2 - 4\pi r^2 \qquad [\because\ S = 6x^2 + 4\pi r^2]$$

$$\Rightarrow \quad x^2 = 9r^2 \quad \Rightarrow \quad x = 3r$$

i.e., x is equal to three times the radius of sphere.

Now, minimum value of V (sum of volume) $= \frac{2}{3}\left\{x^3 + 2\pi\left(\frac{x}{3}\right)^3\right\}$

$$= \frac{2}{3}\left\{x^3 + \frac{2\pi}{27}x^3\right\} = \frac{2}{81}x^3(27 + 2\pi) \text{ cubic unit.}$$

12. Prove that the surface area of a solid cuboid, of square base and given volume, is minimum when it is a cube. *[CBSE (AI) 2017; (F) 2009; CBSE 2005]*

Sol. Let x be the side of square base of cuboid and other side be y.

Then volume of cuboid with square base, $V = x \cdot x \cdot y = x^2y$

As volume of cuboid is given so volume is taken constant throughout the question, therefore,

$$y = \frac{V}{x^2} \qquad ...(i)$$

In order to show that surface area is minimum when the given cuboid is cube, we have to show $S'' > 0$ and $x = y$.

Let S be the surface area of cuboid, then

$$S = x^2 + xy + xy + xy + xy + x^2 = 2x^2 + 4xy \qquad ...(ii)$$

$$= 2x^2 + 4x.\frac{V}{x^2} \quad \Rightarrow \quad S = 2x^2 + \frac{4V}{x} \qquad ...(iii)$$

$$\Rightarrow \quad \frac{dS}{dx} = 4x - \frac{4V}{x^2} \qquad ...(iv)$$

For maximum/minimum value of S, we have $\frac{dS}{dx} = 0$

$$\Rightarrow \quad 4x - \frac{4V}{x^2} = 0 \Rightarrow 4V = 4x^3 \quad \Rightarrow \quad V = x^3 \qquad ...(v)$$

Putting $V = x^3$ in (i), we have

$$y = \frac{x^3}{x^2} = x$$

Here, $y = x \Rightarrow$ cuboid is a cube.

Differentiating (*iv*) w.r.t x, we get

$$\frac{d^2S}{dx^2} = \left(4 + \frac{8V}{x^3}\right) > 0$$

Hence, surface area is minimum when given cuboid is a cube.

13. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area. *[CBSE Delhi 2014]*

Sol. Let r, h be radius and height of closed right circular cylinder having volume 128π cm^3.

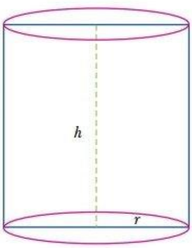
If S be the surface area then

$$S = 2\pi rh + 2\pi r^2 \quad \Rightarrow \quad S = 2\pi(rh + r^2)$$

$$= 2\pi\left(r.\frac{128}{r^2} + r^2\right) \qquad \left[\begin{array}{l}\because \quad V = \pi r^2 h \\ \Rightarrow 128\pi = \pi r^2 h \\ \therefore \quad h = \frac{128}{r^2}\end{array}\right]$$

$$= 2\pi\left(\frac{128}{r} + r^2\right) \quad \Rightarrow \quad \frac{dS}{dr} = 2\pi\left(-\frac{128}{r^2} + 2r\right)$$

For extreme value of S



$$\frac{dS}{dr} = 0 \quad \Rightarrow \quad 2\pi\left(-\frac{128}{r^2} + 2r\right) = 0$$

$$\Rightarrow \quad -\frac{128}{r^2} + 2r = 0$$

$$\Rightarrow \quad 2r = \frac{128}{r^2} \quad \Rightarrow \quad r^3 = \frac{128}{2}$$

$$\Rightarrow \quad r^3 = 64 \quad \Rightarrow \quad r = 4$$

Again $\frac{d^2S}{dr^2} = 2\pi\left(\frac{128 \times 2}{r^3} + 2\right) \Rightarrow \left.\frac{d^2S}{dr^2}\right]_{r=4} = +\text{ve}$

Hence, for $r = 4$ cm, S (surface area) is minimum.

Therefore, dimensions for minimum surface area of cylindrical can are radius $r = 4$ cm and $h = \frac{128}{r^2} = \frac{128}{16} = 8$ cm.

14. If the sum of hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

[CBSE Delhi 2017; (AI) 2009, 2014; (Central) 2016]

Sol. Let h and x be the length of hypotenuse and one side of a right triangle and y is length of the third side.

If A be the area of triangle, then

$$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{h^2 - x^2} \qquad \left[\begin{array}{l}\text{also given} \\ h + x = k \text{ (constant)} \\ \therefore\ h = k - x\end{array}\right]$$

$$A = \frac{1}{2}x\sqrt{(k-x)^2 - x^2} = \frac{1}{2}x\sqrt{k^2 - 2kx + x^2 - x^2}$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}(k^2 - 2kx) \quad \Rightarrow A^2 = \frac{1}{4}(k^2x^2 - 2kx^3)$$

Differentiating with respect to x, we get

$$\frac{d(A^2)}{dx} = \frac{1}{4}(2k^2x - 6kx^2) \qquad \ldots(i)$$

For maxima or minima of A^2

$$\frac{d(A^2)}{dx} = 0 \quad \Rightarrow \quad \frac{1}{4}(2k^2x - 6kx^2) = 0$$

$$\left[\begin{array}{l} \because V = lbh \\ 8 = lb2 \\ \therefore\ b = \frac{8}{2l} = \frac{4}{l} \end{array}\right]$$

$$\Rightarrow \quad 2k^2x - 6kx^2 = 0 \quad \Rightarrow \quad 2kx\,(k - 3x) = 0$$

$$\Rightarrow \quad k - 3x = 0\ ;\ 2kx \neq 0 \quad \Rightarrow \quad x = \frac{k}{3}$$

Differentiating (*i*) again with respect to x, we get

$$\frac{d^2(A^2)}{dx^2} = \frac{1}{4}(2k^2 - 12kx)$$

$$\left.\frac{d^2(A^2)}{dx^2}\right]_{x=k/3} = \frac{1}{4}\left(2k^2 - 12k.\frac{k}{3}\right) = -\frac{k^2}{2} < 0$$

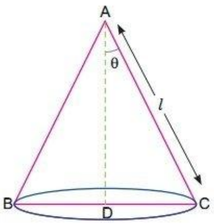
Hence, A^2 is maximum when $x = \frac{k}{3}$ and $h = k - \frac{k}{3} = \frac{2k}{3}$.

i.e., A is maximum when $x = \frac{k}{3}, h = \frac{2k}{3}$

$$\therefore\ \cos\theta = \frac{x}{h} = \frac{k}{3} \times \frac{3}{2k} = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

15. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$. **[*CBSE Delhi 2014; (North) 2016*]**

Sol. Let *ABC* be cone having slant height l and semi-vertical angle θ.



If V be the volume of cone then

$$V = \frac{1}{3}.\pi \times DC^2 \times AD = \frac{\pi}{3} \times l^2 \sin^2\theta \times l\cos\theta$$

$$\Rightarrow \quad V = \frac{\pi l^3}{3}\sin^2\theta\cos\theta$$

$$\Rightarrow \quad \frac{dV}{d\theta} = \frac{\pi l^3}{3}[-\sin^3\theta + 2\sin\theta.\cos^2\theta]$$

For maximum value of V., $\frac{dV}{d\theta} = 0$

$$\frac{\pi l^3}{3}[-\sin^3\theta + 2\sin\theta.\cos^2\theta]. = 0$$

$$\Rightarrow \quad -\sin^3\theta + 2\sin\theta.\cos^2\theta = 0 \quad \Rightarrow \quad -\sin\theta\,(\sin^2\theta - 2\cos^2\theta) = 0$$

$$\Rightarrow \quad \sin\theta = 0 \text{ or } 1 - \cos^2\theta - 2\cos^2\theta = 0$$

$$\Rightarrow \quad \theta = 0 \text{ or } 1 - 3\cos^2\theta = 0$$

$$\Rightarrow \quad \theta = 0 \text{ or } \cos\theta = \frac{1}{\sqrt{3}}$$

Now $$\frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\{-3\sin^2\theta.\cos\theta - 4\sin^2\theta.\cos\theta + 2\cos^3\theta\}$$

$$\Rightarrow \quad \frac{d^2V}{d\theta^2} = \frac{\pi l^3}{3}\{-7\sin^2\theta\cos\theta + 2\cos^3\theta\} \quad \Rightarrow \quad \left.\frac{d^2V}{d\theta^2}\right]_{\theta=0} = +\text{ve}$$

and $$\left.\frac{d^2V}{d\theta^2}\right]_{\cos\theta = \frac{1}{\sqrt{3}}} = -\text{ve} \quad \left[\text{Putting } \cos\theta = \frac{1}{\sqrt{3}} \text{ and } \sin\theta = \sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{\sqrt{2}}{\sqrt{3}}\right]$$

Hence, for $\cos\theta = \frac{1}{\sqrt{3}}$ or $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, V is maximum.

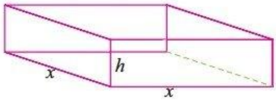
16. An open box with a square base is to be made out of a given quantity of cardboard of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

[*NCERT Exemplar; CBSE (AI) 2012*]

Sol. Let the length, breadth and height of open box with square be x, x and h unit respectively.

If V be the volume of box then $V = x.x.\,h \quad \Rightarrow \quad V = x^2h$ ....(*i*)

Also $\quad c^2 = x^2 + 4xh \quad \Rightarrow \quad h = \frac{c^2 - x^2}{4x}$



Putting it in (*i*), we get

$$V = \frac{x^2(c^2 - x^2)}{4x} \quad \Rightarrow \quad V = \frac{c^2x}{4} - \frac{x^3}{4}$$

Differentiating with respect to x, we get

$$\frac{dV}{dx} = \frac{c^2}{4} - \frac{3x^2}{4}$$

Now for maxima or minima $\frac{dV}{dx} = 0$.

$$\Rightarrow \quad \frac{c^2}{4} - \frac{3x^2}{4} = 0 \quad \Rightarrow \quad \frac{3x^2}{4} = \frac{c^2}{4}$$

$$\Rightarrow \quad x^2 = \frac{c^2}{3} \quad \Rightarrow \quad x = \frac{c}{\sqrt{3}}$$

Now, $\quad \frac{d^2V}{dx^2} = -\frac{6x}{4} = -\frac{3x}{2} \quad \Rightarrow \quad \left[\frac{d^2V}{dx^2}\right]_{x = c/\sqrt{3}} = -\frac{3c}{2\sqrt{3}} = -\text{ve}$

Hence, for $x = \frac{c}{\sqrt{3}}$ volume of box is maximum.

$$\therefore \quad h = \frac{c^2 - x^2}{4x} = \frac{c^2 - \frac{c^2}{3}}{4\frac{c}{\sqrt{3}}} = \frac{2c^2}{3} \times \frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

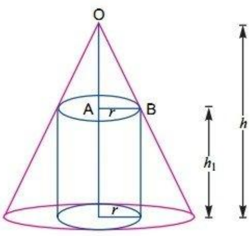
Therefore maximum volume $= x^2 . h = \frac{c^2}{3} . \frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}}$ cubic units

17. Show that the volume of the greatest cylinder that can be inscribed in a cone of height 'h' and semi-vertical angle 'α' is $\frac{4}{27}\pi h^3 \tan^2\alpha$. [*CBSE (AI) 2010, (East) 2016*]

Sol. Let a cylinder of base radius r and height h_1 is included in a cone of height h and semi-vertical angle α.

Then $AB = r$, $OA = (h - h_1)$. In right angle triangle OAB,

$$\frac{AB}{OA} = \tan\alpha \Rightarrow \frac{r}{h - h_1} = \tan\alpha \quad \Rightarrow \quad r = (h - h_1)\tan\alpha$$



$\therefore \; V = \pi\,[(h - h_1)\tan\alpha]^2 . h_1 \qquad (\because \text{ Volume of cylinder} = \pi r^2 h)$

$\quad = \pi\tan^2\alpha . h_1 (h - h_1)^2$ ...(*i*)

Differentiating with respect to h_1, we get

$$\frac{dV}{dh_1} = \pi\tan^2\alpha[h_1 . 2(h - h_1)(-1) + (h - h_1)^2 \times 1]$$

$$= \pi\tan^2\alpha\,(h - h_1)\,[-2h_1 + h - h_1]$$

$$= \pi\tan^2\alpha\,(h - h_1)\,(h - 3h_1)$$

For maximum volume $V, \frac{dV}{dh_1} = 0$

$\Rightarrow \quad h - h_1 = 0$ or $h - 3h_1 = 0 \Rightarrow h = h_1$ or $h_1 = \frac{1}{3}h$

$\Rightarrow \quad h_1 = \frac{1}{3}h \qquad (\because h = h_1$ is not possible)

Again differentiating with respect to h_1, we get

$$\frac{d^2V}{dh_1^2} = \pi \tan^2\alpha[(h - h_1)(-3) + (h - 3h_1)(-1)]$$

At $h_1 = \frac{1}{3}h$, $\frac{d^2V}{dh_1^2} = \pi \tan^2\alpha\left[\left(h - \frac{1}{3}h\right)(-3) + 0\right] = -2\pi h \tan^2\alpha < 0$

$\therefore$ Volume is maximum for $h_1 = \frac{1}{3}h$.

$$V_{max} = \pi \tan^2\alpha.\left(\frac{1}{3}h\right)\left(h - \frac{1}{3}h\right)^2 \qquad \text{[Using } (i)]$$

$$= \frac{4}{27}\pi h^3 \tan^2\alpha$$

18. The sum of the perimeter of a circle and a square is *k*, where *k* is some constant. Prove that the sum of their areas is least when the side of the square is double the radius of the circle.

[*CBSE (F) 2010, 2014*]

Sol. Let side of square be a units and radius of circle be r units.

It is given that $4a + 2\pi r = k$, where k is a constant.

$$\Rightarrow \quad r = \frac{k - 4a}{2\pi}$$

Sum of areas, $A = a^2 + \pi r^2 = a^2 + \pi\left[\frac{k - 4a}{2\pi}\right]^2 = a^2 + \frac{1}{4\pi}(k - 4a)^2$

Differentiating with respect to a, we get

$$\frac{dA}{da} = 2a + \frac{1}{4\pi}.2(k - 4a).(-4) = 2a - \frac{2(k - 4a)}{\pi} \qquad \ldots(i)$$

For minimum area, $\frac{dA}{da} = 0$

$$\Rightarrow \quad 2a - \frac{2(k - 4a)}{\pi} = 0 \Rightarrow 2a = \frac{2(k - 4a)}{\pi}$$

$$\Rightarrow \quad 2a = \frac{2(2\pi r)}{\pi} \qquad [\text{As } k = 4a + 2\pi r \text{ given}]$$

$$\Rightarrow \quad a = 2r$$

Now, again differentiating equation (*i*) with respect to a, we get

$$\frac{d^2A}{da^2} = 2 - \frac{2}{\pi}(-4) = 2 + \frac{8}{\pi} \quad \text{at } a = 2\pi, \quad \frac{d^2A}{da^2} = 2 + \frac{8}{\pi} > 0$$

$\therefore$ For $ax = 2r$, sum of areas is least.

Hence, sum of areas is least when side of the square is double the radius of the circle.

19. Tangent to the circle $x^2 + y^2 = 4$ at any point on it in the first quadrant makes intercepts *OA* and *OB* on *x* and *y* axes respectively, *O* being the centre of the circle. Find the minimum value of (*OA* + *OB*).

[*CBSE Ajmer 2015*]

Sol. Let AB be the tangent in the first quadrant to the circle $x^2 + y^2 = 4$ which make intercepts OA and OB on x and y axis respectively. Let $S = OA + OB$.

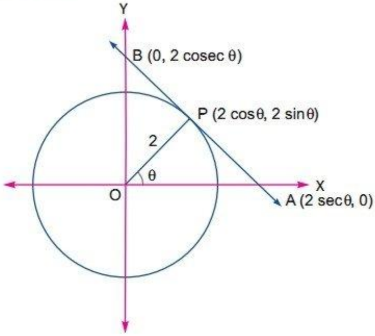
$$S = OA + OB \qquad \ldots (i)$$

Let θ be the angle made by OP with positive direction of x-axis.

$\therefore$ Coordinates of $P = (2\cos\theta, 2\sin\theta)$

Coordinates of $A = (2\sec\theta, 0)$

Coordinates of $B = (0, 2\,\text{cosec}\,\theta)$



$(i) \Rightarrow S = 2\sec\theta + 2\,\text{cosec}\,\theta$

$\Rightarrow \dfrac{dS}{d\theta} = 2\{\sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta\}$

For extremum value of V

$\Rightarrow \dfrac{dS}{d\theta} = 0 \Rightarrow 2\{\sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta\} = 0$

$\Rightarrow \sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta = 0$

$\Rightarrow \dfrac{1}{\cos\theta}\dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\sin\theta}\dfrac{\cos\theta}{\sin\theta} \Rightarrow \dfrac{\sin\theta}{\cos^2\theta} = \dfrac{\cos\theta}{\sin^2\theta}$

$\Rightarrow \sin^3\theta = \cos^3\theta \qquad \Rightarrow \qquad \sin\theta = \cos\theta$

$\Rightarrow \theta = \frac{\pi}{4}\left[\because \theta \text{ lies in first quadrant } \Rightarrow 0 \le \theta \le \frac{\pi}{4}\right]$

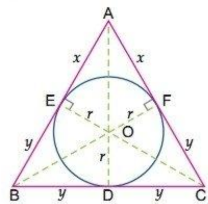
Now, $\dfrac{d^2S}{d\theta^2} = 2\{(\sec^3\theta + \tan^2\theta\sec\theta) + (\text{cosec}^3\theta + \text{cosec}\,\theta\cot^2\theta)\}$

$\Rightarrow \left.\dfrac{d^2S}{d\theta^2}\right]_{\theta=\frac{\pi}{4}} = +\text{ve} \quad \Rightarrow \quad S$ is minimum when $\theta = \dfrac{\pi}{4}$

$\therefore$ Minimum value of $S = OA + OB$ is $2\sec\frac{\pi}{4} + 2\,\text{cosec}\frac{\pi}{4} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$ units.

20. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$. ***[CBSE (Central) 2016]***

Sol. Let ΔABC be isosceles triangle having $AB = AC$ in which a circle with centre O and radius r is inscribed touching sides AB, BC and AC at E, D and F respectively.



Let $AE = AF = x$, $BE = BD = y$

Obviously, $CF = CD = y$

Let P be the perimeter of ΔABC.

$\therefore P = 2x + 4y \Rightarrow P = \dfrac{4yr^2}{y^2 - r^2} + 4y$ (From (i))

Differentiating w.r.t. y, we get

$\Rightarrow \dfrac{dP}{dy} = \dfrac{(y^2 - r^2).4r^2 - 4yr^2(2y - 0)}{(y^2 - r^2)^2} + 4$

$\Rightarrow \dfrac{dP}{dy} = \dfrac{4y^2r^2 - 4r^4 - 8y^2r^2}{(y^2 - r^2)^2} + 4$

$\Rightarrow \dfrac{dP}{dy} = \dfrac{-4r^2(r^2 + y^2)}{(y^2 - r^2)^2} + 4$

For critical point $\dfrac{dP}{dy} = 0$

$\Rightarrow \dfrac{-4r^2(r^2 + y^2)}{(y^2 - r^2)^2} + 4 = 0$

$\Rightarrow -4r^2(r^2 + y^2) + 4(y^2 - r^2)^2 = 0$

$\Rightarrow -r^4 - r^2y^2 + y^4 + r^4 - 2y^2r^2 = 0$

$ar(\Delta ABC) = ar(\Delta BOC) + ar(\Delta AOC) + ar(\Delta AOB)$

$\Rightarrow \frac{1}{2}AD.BC = \frac{1}{2}.BC.OD + \frac{1}{2}.AC.OF + \frac{1}{2}.AB.OE$

$\Rightarrow 2y.(r + \sqrt{r^2 + x^2}) = 2y.r + (x + y).r + (x + y).r$

$\Rightarrow 2y.(r + \sqrt{r^2 + x^2}) = 2yr + 2(x + y).r$

$\Rightarrow yr + y\sqrt{r^2 + x^2} = yr + xr + yr$

$\Rightarrow y\sqrt{r^2 + x^2} = xr + yr$

$\Rightarrow y^2(r^2 + x^2) = x^2r^2 + y^2r^2 + 2xyr^2$

$\Rightarrow y^2r^2 + x^2y^2 = x^2r^2 + y^2r^2 + 2xyr^2$

$\Rightarrow x^2y^2 = x^2r^2 + 2xyr^2$

$\Rightarrow xy^2 = xr^2 + 2yr^2$

$\Rightarrow x = \dfrac{2yr^2}{(y^2 - r^2)} \qquad \ldots(i)$

$\Rightarrow \quad y^4 - 3r^2y^2 = 0$

$\Rightarrow \quad y^2[y^2 - 3r^2] = 0$

$\Rightarrow \quad y = \sqrt{3}r \quad [\because y \neq 0]$

Also $\left.\frac{d^2P}{dr^2}\right]_{\sqrt{3}r} = +\text{ve}$

$\Rightarrow$ when $y = \sqrt{3}r$, the value of P is minimum.

$\therefore$ Least perimeter $= 4y + \frac{4r^2y}{y^2 - r^2} = 4\sqrt{3}r + \frac{4r^2.\sqrt{3}r}{3r^2 - r^2} = 4\sqrt{3}r + \frac{4\sqrt{3}r^3}{2r^2} = 6\sqrt{3}r$ units

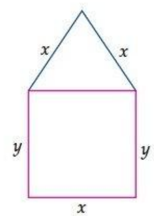
21. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window. *[CBSE (AI) 2011]*

Sol. Let x and y be the dimensions of rectangular part of window and x be side of equilateral part.

If A be the total area of window, then $A = x.y + \frac{\sqrt{3}}{4}x^2$ ...(*i*)

Also, $x + 2y + 2x = 12$

$\Rightarrow \quad 3x + 2y = 12 \quad \Rightarrow \quad y = \frac{12 - 3x}{2}$



$\therefore \quad A = x.\frac{(12 - 3x)}{2} + \frac{\sqrt{3}}{4}x^2$ [From (*i*)]

$\Rightarrow \quad A = 6x - \frac{3x^2}{2} + \frac{\sqrt{3}}{4}x^2$

$\Rightarrow \quad A' = 6 - 3x + \frac{\sqrt{3}}{2}x$ [Differentiating with respect to x]

Now, for maxima or minima

$$A' = 0 \Rightarrow 6 - 3x + \frac{\sqrt{3}}{2}x = 0 \qquad \Rightarrow \qquad x = \frac{12}{6 - \sqrt{3}}$$

Again, $A'' = -3 + \frac{\sqrt{3}}{2} < 0$ (for any value of x) $\Rightarrow A'']_{x = \frac{12}{6-\sqrt{3}}} < 0$ *i.e.*, is maximum if

$$x = \frac{12}{6 - \sqrt{3}} \text{ and } y = \frac{12 - 3\left(\frac{12}{6 - \sqrt{3}}\right)}{2}.$$

i.e., for largest area of window, dimensions of rectangle are

$$x = \frac{12}{6 - \sqrt{3}} \text{ and } y = \frac{18 - 6\sqrt{3}}{6 - \sqrt{3}}.$$

22. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *[CBSE (AI) 2013]*

Sol. Let $ABCD$ be rectangle having area A inscribed in an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(i)$$

Let the coordinate of A be (α, β).

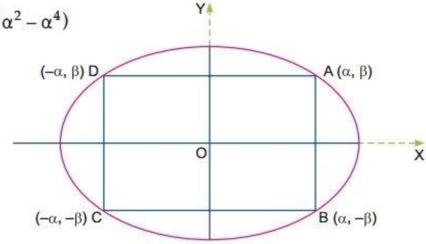
$\therefore$ Coordinate of $B \equiv (\alpha, -\beta)$, $C \equiv (-\alpha, -\beta)$, $D \equiv (-\alpha, \beta)$

Now $A = \text{Length} \times \text{Breadth} = 2\alpha \times 2\beta = 4\alpha\beta$

$$= 4\alpha.\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)}$$

$$\left[\begin{array}{l}\because (\alpha,\beta) \text{ lies on ellipse } (i) \\ \therefore \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1 \text{ i.e., } \beta=\sqrt{b^2\left(1-\frac{\alpha^2}{a^2}\right)}\end{array}\right]$$

$$\Rightarrow \quad A^2 = 16\alpha^2\left\{b^2\left(1-\frac{\alpha^2}{a^2}\right)\right\} \Rightarrow A^2 = \frac{16b^2}{a^2}(a^2\alpha^2-\alpha^4)$$



$$\Rightarrow \quad \frac{d(A^2)}{d\alpha} = \frac{16b^2}{a^2}(2a^2\alpha - 4\alpha^3)$$

For maximum or minimum value

$$\frac{d(A^2)}{d\alpha} = 0$$

$$\Rightarrow \quad 2a^2\alpha - 4\alpha^3 = 0 \quad \Rightarrow \quad 2\alpha(a^2-2\alpha^2) = 0$$

$$\Rightarrow \quad \alpha = 0, \alpha = \frac{a}{\sqrt{2}}$$

Again $\dfrac{d^2(A^2)}{d\alpha^2} = \dfrac{16b^2}{a^2}(2a^2 - 12\alpha^2)$

$$\Rightarrow \quad \left.\frac{d^2(A^2)}{d\alpha^2}\right]_{\alpha=\frac{a}{\sqrt{2}}} = \frac{16b^2}{a^2}\left(2a^2 - 12\times\frac{a^2}{2}\right) < 0$$

$\Rightarrow$ For $\alpha = \dfrac{a}{\sqrt{2}}$, A^2 *i.e.*, A is maximum.

i.e., for greatest area A, $\alpha = \dfrac{a}{\sqrt{2}}$ and $\beta = \dfrac{b}{\sqrt{2}}$ (Using (*i*))

$\therefore$ Greatest area $= 4\alpha\,.\,\beta = 4\dfrac{a}{\sqrt{2}} \times \dfrac{b}{\sqrt{2}} = 2ab$

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius a is $\frac{8}{27}$ of the volume of the sphere. **[*CBSE Delhi 2016; (AI) 2014; (F) 2013*]**

Sol. Consider a sphere of radius a with centre at O such that $OD = x$ and $DC = r$.

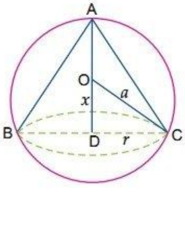
Let h be the height of the cone.

Then $h = AD = AO + OD = a + x$ ...(*i*)

($OA = OC$ = radius)

In the right angle ΔODC,

$a^2 = r^2 + x^2$ (By Pythagoras theorem) ...(*ii*)



Let V be the volume the cone, then $V = \dfrac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V(x) = \frac{1}{3}\pi(a^2-x^2)(a+x) \qquad [\text{From } (i) \text{ and } (ii)]$$

$$\Rightarrow \quad V'(x) = \frac{1}{3}\pi\left[(a^2-x^2)\frac{d}{dx}(a+x) + (a+x)\frac{d}{dx}(a^2-x^2)\right]$$

$$= \frac{1}{3}\pi[(a^2-x^2)(1) + (a+x)(-2x)]$$

$$= \frac{1}{3}\pi[(a+x)(a-x-2x)] = \frac{1}{3}\pi(a+x)(a-3x)$$

Also, $V''(x) = \frac{1}{3}\pi\left[(a+x)\frac{d}{dx}(a-3x) + (a-3x)\frac{d}{dx}(a+x)\right]$

$\Rightarrow$ $V''(x) = \frac{1}{3}\pi[(a+x)(-3) + (a-3x)(1)]$

For maximum or minimum value, we have $V'(x) = 0$.

$$\frac{1}{3}\pi(a+x)(a-3x) = 0 \quad \Rightarrow \quad x = -a \text{ or } x = \frac{a}{3}$$

Neglecting $x = -a$ $[\because x > 0]$

$$V''\left(\frac{a}{3}\right) = \frac{1}{3}\pi\left[\left(a+\frac{a}{3}\right)(-3) + \left(a - 3\left(\frac{a}{3}\right)\right)\right] = \frac{-4\pi a}{3} < 0$$

$\therefore$ Volume is maximum when $x = \frac{a}{3}$.

Putting $x = \frac{a}{3}$ in equation (i) and (ii), we get

$$h = a + \frac{a}{3} = \frac{4a}{3} \quad \text{and} \quad r^2 = a^2 - \frac{a^2}{9} = \frac{8a^2}{9}$$

Now, volume of cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{8a^2}{9}\right)\left(\frac{4a}{3}\right) = \frac{8}{27}\left(\frac{4}{3}\pi a^3\right)$

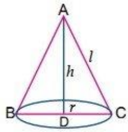
Thus, volume of the cone $= \frac{8}{27}$ (volume of the sphere).

24. Show that the right-circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. [*CBSE (AI) 2011*] [*HOTS*]

Sol. Let ABC be right-circular cone having radius 'r' and height 'h'. If V and S are its volume and surface area (curved) respectively, then

$$S = \pi r l$$

$$S = \pi r\sqrt{h^2 + r^2} \qquad \ldots(i)$$



Also, $V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2}$

Putting the value of h in (i), we get

$$S = \pi r\sqrt{\frac{9V^2}{\pi^2 r^4} + r^2}$$

$\Rightarrow$ $S^2 = \pi^2 r^2\left(\frac{9V^2 + \pi^2 r^6}{\pi^2 r^4}\right)$ [Maxima or Minima is same for S or S^2]

$\Rightarrow$ $S^2 = \frac{9V^2}{r^2} + \pi^2 r^4$

Differentiating with respect to 'r', we get

$\Rightarrow$ $(S^2)' = \frac{-18V^2}{r^3} + 4\pi^2 r^3$ $\qquad \ldots(ii)$

Now, for max. or min. $(S^2)' = 0$

$\Rightarrow$ $-18\frac{V^2}{r^3} + 4\pi^2 r^3 = 0 \quad \Rightarrow \quad 4\pi^2 r^6 = 18V^2$

Putting value of V

$\Rightarrow \quad 4\pi^2 r^6 = 18 \times \frac{1}{9}\pi^2 r^4 h^2 \quad \Rightarrow \quad 2r^2 = h^2 \quad \Rightarrow \quad r = \frac{h}{\sqrt{2}}$

Differentiating (*ii*) with respect to 'r', again

$$(S^2)'' = \frac{54V^2}{r^4} + 12\pi^2 r^2$$

$\Rightarrow \quad (S^2)'']_{r=\frac{h}{\sqrt{2}}} > 0$ (For any value of r)

Hence, S^2 *i.e.*, S is minimum for $r = \frac{h}{\sqrt{2}}$ or $h = \sqrt{2}r$

i.e., for least curved surface, altitude is equal to $\sqrt{2}$ times the radius of the base.

25. Find the maximum area of the isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of major axis. *[CBSE Bhubaneshwar 2015, (AI) 2008]*

Sol. Let ΔABC be an isosceles triangle inscribed in the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

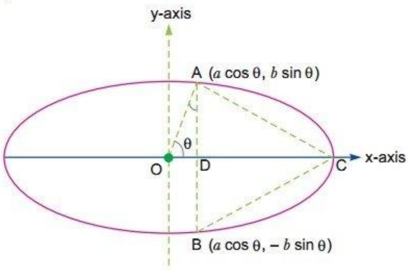
Such that 'C' lies on end of major axis and $AC = BC$.

Let coordinates of A and B be $(a \cos \theta, b \sin \theta)$ and $(a \cos \theta, -b \sin \theta)$ respectively.

If 'A' be the area of inscribed triangle then

$$A = \frac{1}{2} \times AB \times CD = \frac{1}{2} \times 2b \sin\theta \times (a - a\cos\theta)$$

$$= ab \sin \theta (1 - \cos \theta)$$



Differentiating with respect to θ, we get

$$\frac{dA}{d\theta} = ab[\sin\theta.\sin\theta + (1 - \cos\theta).\cos\theta]$$

$$= ab(\sin^2\theta + \cos\theta - \cos^2\theta)$$

For maxima and minima $\frac{dA}{d\theta} = 0$

$\Rightarrow \quad ab(\sin^2\theta + \cos\theta - \cos^2\theta) = 0$

$\Rightarrow \quad \cos\theta - \cos 2\theta = 0$

$\Rightarrow \quad \cos 2\theta = \cos\theta$

$\Rightarrow \quad 2\theta = 2n\pi \pm \theta \quad [\because \cos\theta = \cos\alpha;\ \theta = 2n\pi \pm \alpha]$

$\Rightarrow \quad \theta = n\pi + \frac{\theta}{2}$ or $n\pi - \frac{\theta}{2}$, where $n = 0, \pm 1, \pm 2, ...$

$\Rightarrow \quad \theta = \frac{2\pi}{3} \in (0, \pi)$

$\therefore \quad \frac{d^2A}{d\theta^2} = ab(2\sin\theta.\cos\theta - \sin\theta + 2\cos\theta.\sin\theta) = ab(2\sin 2\theta - \sin\theta)$

$$\left[\frac{d^2A}{d\theta^2}\right]_{\theta=\frac{2\pi}{3}} < 0$$

Hence, for $\theta = \frac{2\pi}{3}$, A is maximum.

Hence, maximum area of triangle $A = ab\sin\frac{2\pi}{3}.\left(1 - \cos\frac{2\pi}{3}\right) = ab\frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}ab$ sq units.

Questions for Practice

■ Objective Type Questions

1. Choose and write the correct option in each of the following questions.

(*i*) The rate of change in volume of sphere wrt r when radius is $\left(\frac{1}{2}\right)$ is

(*a*) π (*b*) 2π (*c*) 4π (*d*) none of these

(*ii*) The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is

(*a*) $10\text{ cm}^2/\text{s}$ (*b*) $\sqrt{3}\text{ cm}^2/\text{s}$ (*c*) $10\sqrt{3}\text{ cm}^2/\text{s}$ (*d*) $\frac{10}{3}\text{ cm}^2/\text{s}$

(*iii*) The maximum value of slope of the curve $y = -x^3 + 3x^2 + 12x - 5$ is *[CBSE 2020 (65/3/1)]*

(*a*) 15 (*b*) 12 (*c*) 9 (*d*) 0

(*iv*) If the function $f(x) = 2x^2 - kx + 5$ is increasing on $[1, 2]$, then k lies in the interval

(*a*) $(-\infty, 4)$ (*b*) $(4, \infty)$ (*c*) $(-\infty, 8)$ (*d*) $(8, \infty)$

(*v*) The function $f(x) = e^x$

(*a*) has no maximum value (*b*) has a maximum value
(*c*) has a stationary point (*d*) None of these

(*vi*) The stationary points of $\sin x$ in $[0, \pi]$ is

(*a*) 0 (*b*) $\frac{\pi}{2}$ (*c*) π (*d*) none of these

(*vii*) For which value of a is the function $f(x) = ax^2 + 2$ decreasing in $[1, 2]$?

(*a*) $(1, 2)$ (*b*) $[1, 2]$ (*c*) $(-\infty, 0)$ (*d*) none of these

(*viii*) The rate of change in area of square w.r.t. side when side is $\frac{1}{\sqrt{3}}$ unit is

(*a*) $\frac{2}{\sqrt{3}}\text{ unit}^2/\text{sec}$ (*b*) $\frac{1}{\sqrt{53}}\text{ unit}^2/\text{sec}$ (*c*) $2\text{ unit}^2/\text{sec}$ (*d*) none of these

■ Conceptual Questions

2. If the rate of change of volume of a sphere is equal to the rate of change of its radius, find the radius of the sphere.
3. Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing. *[CBSE 2020 (65/3/1)]*
4. It is given that at $x = 1$ the function $x^4 - 62x^2 + ax + 9$ attains the maximum value on the interval $[0, 2]$. Find the value of a.

■ Very Short Answer Questions

5. The contentment obtained after eating x-units of a new dish at a trial function is given by the function $C(x) = x^3 + 6x^2 + 5x + 3$. If the marginal contentment is defined as rate of change of $C(x)$ with respect to the number of units consumed at an instant, then find the marginal contentment when three units of dish are consumed. *[CBSE (F) 2013]*

6. Prove that the function $f(x) = \tan x - 4x$ is strictly decreasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.

7. Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \frac{\pi}{6}$.

8. Show that the function f defined by $f(x) = (x - 1)e^x + 1$ is an increasing function for all $x > 0$.
[CBSE 2020 (65/4/1)]

■ Short Answer Questions

9. Let 'a' be a real number such that the function $f(x) = ax^2 + 6x - 15$, $x \in \mathbb{R}$ is increasing in $\left(-\infty, \frac{3}{4}\right)$ and decreasing in $\left(\frac{3}{4}, \infty\right)$. Find the point where the function $g(x) = ax^2 - 6x + 15$, $x \in \mathbb{R}$ has local maxima.

10. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is
(*a*) strictly increasing (*b*) strictly decreasing. [CBSE Delhi 2014]

11. Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. [CBSE Delhi 2014]

12. Find all the points of local maxima and local minima of the function

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$$

■ Long Answer Questions

13. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its side. Also, find the maximum volume.
[CBSE 2020 (65/4/1)]

14. Show that a right circular cylinder of the given volume open at the top has minimum total surface area, provided its height is equal to the radius of the base. [CBSE (F) 2014]

15. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm. [CBSE Delhi 2017]

16. Find the local maxima and local minima, of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$, Also find the local maximum and local minimum values. [CBSE Delhi 2015]

17. Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \cos^2 x + \sin x$, $x \in [0, \pi]$. [CBSE Guwahati 2015]

18. Determine the intervals in which the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is strictly increasing or strictly decreasing. [CBSE (South) 2016]

19. A manufacturer can sell x items at a price of ₹$\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is ₹$\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit. [CBSE (AI) 2009]

20. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of the two pieces, so that the combined area of the square and the rectangle is minimum?
[CBSE (F) 2017]

21. Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius r is the square of side $r\sqrt{2}$. **[*CBSE Delhi 2011*]**

22. Show that the rectangle of maximum area that can be inscribed in a circle is a square. **[*CBSE Delhi 2008, 2011*]**

Answers

1. (*i*) (*a*) (*ii*) (*c*) (*iii*) (*a*) (*iv*) (*a*) (*v*) (*a*) (*vi*) (*b*) (*vii*) (*c*) (*viii*) (*a*)

2. $\frac{1}{2\sqrt{\pi}}$ units 3. $(-\infty, -2)$ 4. $a = 120$ 5. 68 units 9. $\left(\frac{-3}{4}, \frac{69}{4}\right)$

10. (*a*) $(-1, 0) \cup (2, \infty)$ (*b*) $(-\infty, -1) \cup (0, 2)$ 12. Local maxima at 0, –5; and local minima at –3

13. Length = 12 cm, breadth = 6 cm and maximum volume = $\frac{216}{\pi}$ cm^3 15. 3 cm^2/sec

16. Local maximum value = $\sqrt{2}$, local minimum value = $-\sqrt{2}$

17. Absolute maximum value = $\frac{5}{4}$ at $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$, absolute minimum value = 1 at $x = 0, \frac{\pi}{2}$ and π

18. $(1, 2) \cup (3, \infty)$; $(-\infty, 1) \cup (2, 3)$ 19. 240 items 20. 16 m, 18 m