Continuity and Differentiability

Multiple Choice Questions

Choose and write the correct option in the following questions.

1. The value of k ($k < 0$) for which the function f defined as

$$f(x) = \begin{cases} \dfrac{1 - \cos kx}{x \sin x}, & x \neq 0 \\ \dfrac{1}{2}, & x = 0 \end{cases}$$

is continuous at $x = 0$ is *[CBSE Sample Paper 2022 (Term-1)]*

(a) ± 1 (b) -1 (c) $\pm \frac{1}{2}$ (d) $\frac{1}{2}$

2. If the function f defined by $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x} & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$, is continuous at $x = \frac{\pi}{2}$, then the value of k is *[CBSE 2021-22 (65/2/4) (Term-1)]*

(a) 2 (b) 3 (c) 6 (d) -6

3. The function $f(x) = \cot x$ is discontinuous on the set *[NCERT Exemplar]*

(a) $\{x = n\pi : n \in Z\}$ (b) $\{x = 2n\pi : n \in Z\}$

(c) $\left\{x = (2n+1)\frac{\pi}{2}; n \in Z\right\}$ (d) $\left\{x = \frac{n\pi}{2}; n \in Z\right\}$

4. The function $f(x) = \dfrac{x - 1}{x(x^2 - 1)}$ is discontinuous at *[CBSE 2020 (65/2/2)]*

(a) exactly one point (b) exactly two points

(c) exactly three points (d) no point

5. The function $f(x) = \begin{cases} \dfrac{e^{3x} - e^{-5x}}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$, is continuous at $x = 0$ for the value of k, as

[CBSE 2021-22 (65/4/2) (Term-1)]

(a) 3 (b) 5 (c) 2 (d) 8

6. The point (s), at which the function f given by $f(x) = \begin{cases} \dfrac{x}{|x|}, & x < 0 \\ -1, & x \geq 0 \end{cases}$ is continuous, is/are

[CBSE Sample Paper 2021-22 (Term-1)]

(a) $x \in R$ (b) $x = 0$ (c) $x \in R - \{0\}$ (d) $x = -1$ and 1

7. The value of k for which $f(x) = \begin{cases} 3x + 5, & x \geq 2 \\ kx^2, & x < 2 \end{cases}$ is a continuous function, is

[CBSE 2023 (65/1/1)]

(a) $-\frac{11}{4}$ (b) $\frac{4}{11}$ (c) 11 (d) $\frac{11}{4}$

8. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer function, is continuous at

[CBSE 2022 (Term-1), 2023 (65/5/1)]

(a) 4 (b) -2 (c) 1 (d) 1.5

9. **The function $f(x) = |x|$ is** **[CBSE 2023 (65/4/1)]**

(a) continuous and differentiable everywhere.

(b) continuous and differentiable nowhere.

(c) continuous everywhere, but differentiable everywhere except at $x = 0$.

(d) continuous everywhere, but differentiable nowhere.

10. **The function $f(x) = x|x|$ is** **[CBSE 2023 (65/2/1)]**

(a) continuous and differentiable at $x = 0$.

(b) continuous but not differentiable at $x = 0$.

(c) differentiable but not continuous at $x = 0$.

(d) neither differentiable nor continuous at $x = 0$.

11. **The function $f(x) = e^{|x|}$ is** **[NCERT Exemplar]**

(a) continuous everywhere but not differentiable at $x = 0$

(b) continuous and differentiable everywhere

(c) not continuous at $x = 0$

(d) none of these

12. **The function $f: R \to R$ given by $f(x) = -|x-1|$ is** **[CBSE 2020 (65/2/1)]**

(a) continuous as well as differentiable at $x = 1$

(b) not continuous but differentiable at $x = 1$

(c) continuous but not differentiable at $x = 1$

(d) neither continuous nor differentiable at $x = 1$

13. **The function $f(x) = \begin{cases} x^2 & \text{for } x < 1 \\ 2-x & \text{for } x \geq 1 \end{cases}$ is** **[CBSE 2021-22 (Term-1)]**

(a) not differentiable at $x = 1$

(b) differentiable at $x = 1$

(c) not continuous at $x = 1$

(d) neither continuous nor differentiable at $x = 1$

14. **The value of k for which function $f(x) = \begin{cases} x^2, & x \geq 0 \\ kx, & x < 0 \end{cases}$ is differentiable at $x = 0$ is** **[CBSE 2023 (65/3/2)]**

(a) 1 (b) 2 (c) any real number (d) 0

15. **If $(x^2 + y^2)^2 = xy$, then $\dfrac{dy}{dx}$ is** **[CBSE 2021-22 (65/2/4) (Term-1)]**

(a) $\dfrac{y + 4x(x^2+y^2)}{4y(x^2+y^2) - x}$ (b) $\dfrac{y - 4x(x^2+y^2)}{x + 4(x^2+y^2)}$ (c) $\dfrac{y - 4x(x^2+y^2)}{4y(x^2+y^2) - x}$ (d) $\dfrac{4y(x^2+y^2) - x}{y - 4x(x^2+y^2)}$

16. **If $\sin y = x \cos(a + y)$, then $\dfrac{dx}{dy}$ is** **[CBSE 2021-22 (65/2/4) (Term-1)]**

(a) $\dfrac{\cos a}{\cos^2(a+y)}$ (b) $\dfrac{-\cos a}{\cos^2(a+y)}$ (c) $\dfrac{\cos a}{\sin^2 y}$ (d) $\dfrac{-\cos a}{\sin^2 y}$

17. **Differential of $\log[\log(\log(x^5)]$ wrt x, is** **[CBSE 2021-22 (65/2/4) (Term-1)]**

(a) $\dfrac{5}{x\log(x^5)\log(\log x^5)}$

(b) $\dfrac{5}{x\log(\log x^5)}$

(c) $\dfrac{5x^4}{\log(x^5)\log(\log x^5)}$

(d) $\dfrac{5x}{\log(x^5)\log(\log x^5)}$

18. **If $x = A\cos 4t + B\sin 4t$, then $\dfrac{d^2x}{dt^2}$ is equal to** **[CBSE 2023 (65/5/1)]**

(a) x (b) $-x$ (c) $16x$ (d) $-16x$

19. If $\tan\left(\frac{x+y}{x-y}\right)=k$, then $\frac{dy}{dx}$ is equal to [CBSE 2023 (65/2/1)]

(a) $\frac{-y}{x}$ (b) $\frac{y}{x}$ (c) $\sec^2\left(\frac{y}{x}\right)$ (d) $-\sec^2\left(\frac{y}{x}\right)$

20. If $y=\sin^{-1}x$, then $(1-x^2)\,y_2$ is equal to [CBSE Sample Paper 2023]

(a) xy_1 (b) xy (c) xy_2 (d) x^2

21. If $y=\sin(m\sin^{-1}x)$, then which of the following equations is true? [CBSE 2021-22 (65/2/4) (Term-1)]

(a) $(1-x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+m^2y=0$ (b) $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$

(c) $(1+x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$ (d) $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2x=0$

22. If $y=e^{-x}$, then $\frac{d^2y}{dx^2}$ is equal to [CBSE 2022 (Term-1)]

(a) $-y$ (b) y (c) x (d) $-x$

23. If $y^2(2-x)=x^3$, then $\left(\frac{dy}{dx}\right)_{(1,1)}$ is equal to [CBSE 2022 (Term-1)]

(a) 2 (b) -2 (c) 3 (d) $-\frac{3}{2}$

24. If $y=\frac{\cos x-\sin x}{\cos x+\sin x}$, then $\frac{dy}{dx}$ is [CBSE 2023 (65/3/2)]

(a) $-\sec^2\left(\frac{\pi}{4}-x\right)$ (b) $\sec^2\left(\frac{\pi}{4}-x\right)$ (c) $\log\left|\sec\left(\frac{\pi}{4}-x\right)\right|$ (d) $-\log\left|\sec\left(\frac{\pi}{4}-x\right)\right|$

25. Differential coefficient of $\sec(\tan^{-1}x)$ w.r.t. x is [NCERT Exemplar]

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) $\frac{x}{1+x^2}$ (c) $x\sqrt{1+x^2}$ (d) $\frac{1}{\sqrt{1+x^2}}$

26. If $y=\sin^2(x^3)$, then $\frac{dy}{dx}$ is equal to [CBSE 2023 (65/4/1)]

(a) $2\sin x^3\cos x^3$ (b) $3x^3\sin x^3\cos x^3$ (c) $6x^2\sin x^3\cos x^3$ (d) $2x^2\sin^2(x^3)$

Answers

1. (b)	2. (c)	3. (a)	4. (c)	5. (d)	6. (a)	7. (d)
8. (d)	9. (c)	10. (a)	11. (a)	12. (c)	13. (a)	14. (d)
15. (c)	16. (a)	17. (a)	18. (d)	19. (b)	20. (a)	21. (b)
22. (b)	23. (a)	24. (a)	25. (a)	26. (c)		

Solutions of Selected Multiple Choice Questions

1. $\lim_{x\to 0}\left(\frac{1-\cos kx}{x\sin x}\right)=\frac{1}{2}$ $\Rightarrow$ $\lim_{x\to 0}\left(\frac{2\sin^2\frac{kx}{2}}{x\sin x}\right)=\frac{1}{2}$

$\Rightarrow \lim_{x\to 0} 2\left(\frac{k}{2}\right)^2\left(\frac{\sin\frac{kx}{2}}{\frac{kx}{2}}\right)^2\left(\frac{x}{\sin x}\right)=\frac{1}{2}$

$\Rightarrow k^2=1 \Rightarrow k=\pm 1$ but $k<0 \Rightarrow k=-1$

$\therefore$ Option (b) is correct.

2. Given $f(x)$ is continuous at $x = \frac{\pi}{2}$.

$\therefore \quad \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \left(\frac{k\cos x}{\pi - 2x}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-k\sin x}{-2} = \lim_{x \to \frac{\pi}{2}} \frac{k\sin x}{2}$$

$$= \frac{k}{2}\sin\frac{\pi}{2} = \frac{k}{2}$$

$\because \quad f\left(\frac{\pi}{2}\right) = 3$

$\Rightarrow \quad \lim_{x \to 0} f(x) = f\left(\frac{\pi}{2}\right)$

$\Rightarrow \quad \frac{k}{2} = 3 \Rightarrow k = 6$

$\therefore$ Option (c) is correct.

3. We know that, $f(x) = \cot x$ is continuous in $R - \{n\pi : n \in Z\}$.

Since, $f(x) = \cot x = \frac{\cos x}{\sin x}$ [since, $\sin x = 0$ at $\{n\pi, n \in Z\}$]

Hence, $f(x) = \cot x$ is discontinuous on the set $\{x = n\pi : n \in Z\}$.

$\therefore$ Option (a) is correct.

4. We have,

$$f(x) = \frac{x-1}{x(x^2-1)}$$

$\therefore \quad f(x)$ is discontinuous when $x(x^2 - 1) = 0$.

$\Rightarrow \quad x = 0,\ x = \pm 1$

$\therefore \quad f(x)$ is discontinuous at $x = 0, -1, 1$.

i.e., exactly at three points.

$\therefore$ Option (c) is correct.

5. Given function is continuous at $x = 0$.

$\therefore \quad \lim_{x \to 0} f(x) = f(0)$

$\because \quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{e^{3x} - e^{-5x}}{x}\right)$

$$= \lim_{x \to 0} \left(\frac{3e^{3x} + 5e^{-5x}}{1}\right) = 3 + 5 = 8$$

$\because \quad f(0) = k$

$\Rightarrow \quad k = 8$

$\therefore$ Option (d) is correct.

6. Given, $f(x) = \begin{cases} \frac{x}{-x} = -1, & x < 0 \\ -1, & x \geq 0 \end{cases}$

$\Rightarrow \quad f(x) = -1\ \forall\ x \in R$

$\Rightarrow \quad f(x)$ is continuous $\forall\ x \in R$ as it is a constant function.

$\therefore$ Option (a) is correct.

7. Given function, $f(x)=\begin{cases}3x+5 &, & x \geqslant 2\\ kx^2 &, & x<2\end{cases}$

At $x = 2$

LHL $= \lim_{h \to 0} f(2-h) = \lim_{h \to 0} k(2-h)^2 = k(2-0)^2 = 4k$

RHL $= \lim_{h \to 0} f(2+h) = \lim_{h \to 0} 3(2+h)+5 = 3(2+0)+5 = 11$

and $f(2) = 3 \times 2 + 5 = 11$

Since $f(x)$ is continuous at $x = 2$

$\therefore \quad$ LHL = RHL = $f(2)$

$\Rightarrow \quad 4k = 11$

$\Rightarrow \quad k = \dfrac{11}{4}$

$\therefore$ Option (d) is correct.

8. We have $f(x) = [x]$

At $x = 1.5$

LHL $= \lim_{x \to 1.5^-} f(x) = \lim_{x \to 1.5^-} [x] = 1$

RHL $= \lim_{x \to 1.5^+} f(x) = \lim_{x \to 1.5^+} [x] = 1$

$f(1.5) = [1.5] = 1$

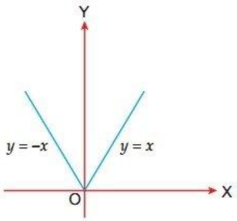
Hence, LHL = RHL = $f(1.5)$

$\Rightarrow$ $f(x)$ is continuous at $x = 1.5$.

$\therefore$ Option (d) is correct.

9. Given function $f(x) = |x|$.

On plotting the curve $f(x) = |x|$, we have



form the graph of $f(x) = |x|$, it is clear that $|x|$ is continuous everywhere, but differentiable everywhere except $x = 0$.

$\therefore$ Option (c) is correct.

10. Given function,

$$f(x) = x|x| = \begin{cases}x^2, & x \geqslant 0\\ -x^2, & x < 0\end{cases}$$

At $x = 0$

LHL $= \lim_{x \to 0^-} -x^2 = -0^2 = 0$

RHL $= \lim_{x \to 0^+} x^2 = 0^2 = 0$

$f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Now, LHD $= \dfrac{d}{dx}(-x^2)\Big|_{x=0} = -2x\big|_{x=0} = -2 \times 0 = 0$

RHD $= \dfrac{d}{dx}(x^2)\Big|_{x=0} = 2x\big|_{x=0} = 2 \times 0 = 0$

$\Rightarrow$ LHD = RHD at $x = 0$

$\Rightarrow$ It is continuous and differentiable at $x = 0$.

$\therefore$ Option (a) is correct.

12. We have,

$$f(x) = -|x-1| = \begin{cases} x-1, & \text{if } x \le 1 \\ -(x-1), & \text{if } x > 1 \end{cases}$$

At $x = 1$

LHL $= \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \{(1-h)-1\} = 0$

RHL $= \lim_{h \to 0} f(1+h) = \lim_{h \to 0} -(1+h-1) = 0$

$f(1) = 1 - 1 = 0$

$\therefore$ LHL = RHL = $f(0)$ $\Rightarrow$ $f(x)$ is continuous everywhere.

Now, at $x = 1$

$$\text{LHD} = \frac{d}{dx}(x-1)\Big|_{x=1} = 1;\ \text{RHD} = \frac{d}{dx}\{-(x-1)\}\Big|_{x=1} = -1$$

$\Rightarrow$ LHD $\neq$ RHD

Hence, $f(x)$ is not differentiable at $x = 1$.

$\therefore$ Option (c) is correct.

13. Given function $f(x) = \begin{cases} x^2 & \text{for} \quad x < 1 \\ 2-x & \text{for} \quad x \geqslant 1 \end{cases}$

At $x = 1$

LHD $= \dfrac{dx^2}{dx}\Big|_{x=1} = 2x|_{x=1} = 2 \times 1 = 2$

RHD $= \dfrac{d(2-x)}{dx}\Big|_{x=1} = -1$

Here, LHD $\neq$ RHD

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

$\therefore$ Option (a) is correct.

At $x = 1$

LHL $= \lim_{x \to 1^-} x^2 = (1)^2 = 1$

RHL $= \lim_{x \to 1^+} 2 - x = 2 - 1 = 1$

$f(1) = 2 - 1 = 1$

LHL = RHL = $f(1)$

$\therefore$ $f(x)$ is continuous.

14. Given function, $f(x) = \begin{cases} x^2, & x \ge 0 \\ kx, & x < 0 \end{cases}$ is differentiable at $x = 0$

At $x = 0$

RHD $= \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} 2x = 2 \times 0 = 0$

LHD $= \lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} k = k$

Since $f(x)$ is differentiable of $x = 0$.

$\Rightarrow$ LHD = RHD $\Rightarrow$ $k = 0$

$\therefore$ Option (d) is correct.

15. $\because$ $(x^2 + y^2)^2 = xy$

Differentiating wrt x, we get

$$2(x^2+y^2)\left(2x + 2y\frac{dy}{dx}\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad 4x(x^2+y^2) + 4y(x^2+y^2)\frac{dy}{dx} = \frac{xdy}{dx} + y$$

$$\Rightarrow \quad \{4y(x^2+y^2) - x\}\frac{dy}{dx} = y - 4x(x^2+y^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$$

$\therefore$ Option (*c*) is correct.

16. $\because \ \sin y = x \cos(a + y)$

$$\Rightarrow \quad \frac{\sin y}{\cos(a+y)} = x \qquad \Rightarrow \qquad x = \frac{\sin y}{\cos(a+y)}$$

$$\therefore \quad \frac{dx}{dy} = \frac{\cos(a+y)\cos y - \sin y \times [-\sin(a+y)]}{\cos^2(a+y)} = \frac{\cos(a+y)\cos y + \sin(a+y)\sin y}{\cos^2(a+y)}$$

$$= \frac{\cos(a+y-y)}{\cos^2(a+y)} = \frac{\cos a}{\cos^2(a+y)}$$

$\therefore$ Option (*a*) is correct.

17. Let $y = \log[\log(\log(x^5))]$

By using chain rule, we have

$$\frac{dy}{dx} = \frac{d}{dx}\log\left[\log\left(\log x^5\right)\right]$$

$$= \frac{1}{\log(\log x^5)} \times \frac{1}{\log(x^5)} \times \frac{1}{x^5} \times 5x^4 = \frac{5}{x\log(x^5)\log(\log x^5)}$$

$\therefore$ Option (*a*) is correct.

18. $x = A\cos 4t + B\sin 4t$

$$\frac{dx}{dt} = -4A\sin 4t + 4B\cos 4t$$

$$\frac{d^2x}{dt^2} = -16A\cos 4t - 16B\sin 4t$$

$$= -16(A\cos 4t + B\sin 4t) = -16x$$

$\therefore$ Option (*d*) is correct.

19. Given, $\tan\left(\frac{x+y}{x-y}\right) = k$

$$\Rightarrow \quad \frac{x+y}{x-y} = \tan^{-1}k$$

Differentiate both sides w.r.t. x, we have

$$\frac{(x-y)\frac{d(x+y)}{dx} - (x+y)\frac{d(x-y)}{dx}}{(x-y)^2} = 0$$

$$\Rightarrow \quad (x-y)\left(1 + \frac{dy}{dx}\right) - (x+y)\left(1 - \frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad (x-y) + (x-y)\frac{dy}{dx} - (x+y) + (x+y)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(x - y + x + y) = 2y \qquad \Rightarrow 2x\frac{dy}{dx} = 2y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

$\therefore$ Option (*b*) is correct.

20. $y = \sin^{-1}x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \sqrt{1-x^2}.\frac{dy}{dx} = 1$$

Again, differentiating both sides with respect to x, we get

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{-2x}{2\sqrt{1-x^2}}=0 \quad\Rightarrow\quad (1-x^2)\frac{d^2y}{dx^2}=xy_1$$

$\Rightarrow\ (1-x^2)y_2 = xy_1$

$\therefore$ Option (a) is correct.

21. Given $y=\sin(m\sin^{-1}x)$

$$\therefore\quad \frac{dy}{dx}=\cos(m\sin^{-1}x)\times\frac{m}{\sqrt{1-x^2}}=\frac{m\cos(m\sin^{-1}x)}{\sqrt{1-x^2}}$$

$$\therefore\quad \frac{d^2y}{dx^2}=m\left\{\frac{\sqrt{1-x^2}\times-[\sin(m\sin^{-1}x)]\times\frac{m}{\sqrt{1-x^2}}-\cos(m\sin^{-1}x)\times\frac{1}{2\sqrt{1-x^2}}\times(-2x)}{\left(\sqrt{1-x^2}\right)^2}\right\}$$

$$=\frac{-m^2\sin(m\sin^{-1}x)+\frac{xm\cos(m\sin^{-1}x)}{\sqrt{1-x^2}}}{1-x^2}$$

$$\Rightarrow\quad (1-x^2)\frac{d^2y}{dx^2}=-m^2\sin(m\sin^{-1}x)+x\frac{dy}{dx}$$

$$\Rightarrow\quad (1-x^2)\frac{d^2y}{dx^2}=-m^2y+x\frac{dy}{dx} \quad\Rightarrow\quad (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$$

$\therefore$ Option (b) is correct.

22. We have, $y=e^{-x}\ \Rightarrow\ \frac{dy}{dx}=-e^{-x}\ \Rightarrow\ \frac{d^2y}{dx^2}=e^{-x}\ \Rightarrow\ \frac{d^2y}{dx^2}=y$

$\therefore$ Option (b) is correct.

23. Given, $y^2(2-x)=x^3\ \Rightarrow\ y^2=\frac{x^3}{2-x}$

Taking log on both sides, we have

$2\log y=3\log x-\log(2-x)$

Differentiating w.r.t. x, we get

$$2\times\frac{1}{y}\frac{dy}{dx}=3\times\frac{1}{x}-\frac{1}{2-x}\times(-1)=\frac{3}{x}+\frac{1}{2-x}$$

$$\frac{dy}{dx}=\frac{y}{2}\left(\frac{3}{x}+\frac{1}{2-x}\right) \quad\Rightarrow\quad \left.\frac{dy}{dx}\right|_{\text{at }(1,1)}=\frac{1}{2}\left(\frac{3}{1}+\frac{1}{2-1}\right)=2$$

$\therefore$ Option (a) is correct.

24. Given, $y=\frac{\cos x-\sin x}{\cos x+\sin x}\ \Rightarrow\ y=\frac{1-\tan x}{1+\tan x}$

(Dividing numerator and denominator by $\cos x$)

$$\Rightarrow\quad y=\tan\left(\frac{\pi}{4}-x\right)\quad\Rightarrow\quad\frac{dy}{dx}=-\sec^2\left(\frac{\pi}{4}-x\right)$$

$\therefore$ Option (a) is correct.

26. Given, $y=\sin^2(x^3)$

$$\therefore\quad\frac{dy}{dx}=\frac{d\sin^2(x^3)}{dx}$$

$$=2\sin(x^3)\times\cos(x^3)\times3x^2=6x^2\sin(x^3)\,.\cos(x^3)$$

$\therefore$ Option (c) is correct.

Assertion-Reason Questions

The following questions consist of two statements—Assertion(A) and Reason(R). Answer these questions selecting the appropriate option given below:

(a) Both A and R are true and R is the correct explanation for A.
(b) Both A and R are true but R is not the correct explanation for A.
(c) A is true but R is false.
(d) A is false but R is true.

1. **Assertion (A) :** If $f(x).g(x)$ is continuous at $x = a$. then $f(x)$ and $g(x)$ are separately continuous at $x = a$.
   Reason (R) : Any function $f(x)$ is said to be continuous at $x = a$, if $\lim_{h \to 0} f(a+h) = f(a)$.
2. **Assertion (A) :** If $f(x)$ and $g(x)$ are two continuous functions such that $f(0) = 3$, $g(0) = 2$, then $\lim_{x \to 0} \{f(x) + g(x)\} = 5$.
   Reason (R) : If $f(x)$ and $g(x)$ are two continuous functions at $x = a$ then $\lim_{x \to a} \{f(x) + g(x)\} = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.
3. **Assertion (A) :** $|\sin x|$ is a continuous function.
   Reason (R) : If $f(x)$ and $g(x)$ both are continuous functions, then $gof(x)$ is also a continuous function.
4. **Assertion (A) :** If $y = \sin x$, then $\frac{d^3y}{dx^3} = -1$ at $x = 0$.
   Reason (R) : If $y = f(x) . g(x)$, then $\frac{dy}{dx} = f(x) . \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$.
5. **Assertion (A) :** If $f(x) = \sin^{-1}x + \cos^{-1}x + 2$ then $f'(1) = 0$.
   Reason (R) : $\frac{d}{dx} \sin x = \cos x$
6. **Assertion (A) :** If $y = \sin^{-1} \frac{2x}{1+x^2}$, then $\frac{dy}{dx} = \frac{2}{1+x^2}$.
   Reason (R) : $\sin^{-1} \frac{2x}{1+x^2} = 2\tan^{-1}x$ and $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$
7. **Assertion (A) :** If $y = (\sin x + \cos x)^2$, then $\left(\frac{dy}{dx}\right)_{\text{at } x = \frac{\pi}{4}} = 0$.
   Reason (R) : $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Answers

1. (*d*)	**2.** (*a*)	**3.** (*a*)	**4.** (*b*)	**5.** (*b*)	**6.** (*a*)	**7.** (*b*)

Solutions of Assertion-Reason Questions

1. Let $f(x) = x$, $g(x) = \text{signum}(x) = \begin{cases} \frac{|x|}{x}, \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases}$, then $G(x) = f(x)\, g(x) = |x|$.

   Which is continuous at $x = 0$.

But $g(x)$ = signum x is not continuous at $x = 0$.

So A is false but R is true.

$\therefore$ Option (d) is correct.

2. $\lim_{x \to 0} \{f(x) + g(x)\} = f(0) + g(0) = 3 + 2 = 5$

So A as well as R is true and R gives correct explanation of statement A.

$\therefore$ Option (a) is correct.

3. We have $f(x) = \sin x$ and $g(x) = |x|$

$\therefore$ $gof(x) = g(f(x)) = g(\sin x) = |\sin x|$

Here, $\sin x$ and $|x|$ both are continuous functions.

$\therefore$ $|\sin x|$ is also continuous function.

Clearly, both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

$\therefore$ Option (a) is correct.

4. We have, $y = \sin x \Rightarrow \dfrac{dy}{dx} = \cos x \Rightarrow \dfrac{d^2y}{dx^2} = -\sin x$

$$\therefore \quad \frac{d^3y}{dx^3} = -\cos x = -1, \text{ at } x = 0$$

Clearly, both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A).

$\therefore$ Option (b) is correct.

5. We have, $f(x) = \sin^{-1}x + \cos^{-1}x + 2$

$\Rightarrow$ $f(x) = \dfrac{\pi}{2} + 2$ $\left[\because \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}\right]$

$\therefore$ $f'(x) = 0 \Rightarrow f'(1) = 0$

Clearly, both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A).

$\therefore$ Option (b) is correct.

6. $y = \sin^{-1}\dfrac{2x}{1+x^2} = 2\tan^{-1}x$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x) = 2\frac{d}{dx}(\tan^{-1}x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+x^2}$$

So A is true statement.

Also R is a correct statement and gives the correct explanation of A.

$\therefore$ Option (a) is correct.

7. $\because$ $y = (\sin x + \cos x)^2$

$$\Rightarrow \frac{dy}{dx} = 2(\sin x + \cos x)(\cos x - \sin x) = 2(\cos^2 x - \sin^2 x)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = 2\left(\cos^2\frac{\pi}{4} - \sin^2\frac{\pi}{4}\right) = 2\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

So A is a correct statement.

Also statement R is correct, but does not give the correct explanation of the statement A.

$\therefore$ Option (b) is correct.

Case-based/Data-based Questions

1. Read the following passage and answer the following questions. *[CBSE 2023 (65/4/1)]*

Let $f(x)$ be a real valued function. Then its

Left Hand Derivative (LHD) : $Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$

Right Hand Derivative (RHD) : $Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its LHD and RHD at $x = a$ exist and both are equal.

For the function $f(x) = \begin{cases} |x-3| & , x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$

(*i*) What is RHD of $f(x)$ at $x = 1$?

(*ii*) What is LHD of $f(x)$ at $x = 1$?

(*iii*) (*a*) Check if the function $f(x)$ is differentiable at $x = 1$.

OR

(*iii*) (*b*) Find $f'(2)$ and $f'(-1)$.

Sol. (*i*) RHD $= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$$Rf'(1) = \lim_{h \to 0} \frac{|1+h-3| - |-2|}{h}$$

$$\Rightarrow \quad Rf'(1) = \lim_{h \to 0} \frac{|-2+h| - 2}{h}$$

$$= \lim_{h \to 0} \frac{-(-2+h) - 2}{h} = \lim_{h \to 0} \frac{2-h-2}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = -1$$

(*ii*) LHD $= \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{\frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4} - |1-3|}{-h}$

$$Lf'(1) = \lim_{h \to 0} \frac{\frac{(1-h)^2}{4} - \frac{3}{2} + \frac{3h}{2} + \frac{13}{4} - 2}{-h}$$

$$\Rightarrow \quad Lf'(1) = \lim_{h \to 0} \frac{(1+h^2-2h) - 6 + 6h + 13 - 8}{-4h} = \lim_{h \to 0} \frac{h^2+4h}{-4h}$$

$$= \lim_{h \to 0} \frac{h(h+4)}{-4h} = \lim_{h \to 0} \frac{h+4}{-4} = \frac{0+4}{-4} = -1$$

(*iii*) (*a*) For the given function, we have at $x = 1$

LHD $= Lf'(1) = -1$

and, RHD $= Rf'(1) = -1$

$\Rightarrow$ LHD = RHD at $x = 1$

$\therefore$ $f(x)$ is differentiable at $x = 1$.

OR

(*iii*) (*b*) We have, $f(x) = |x-3|$ for $x = 2$

$\Rightarrow \quad f(x) = -(x-3) = -x+3$

$\Rightarrow \quad f'(x) = -1$

$\Rightarrow \quad f'(2) = -1$

and, $\quad f(x) = \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}$ for $x = -1$

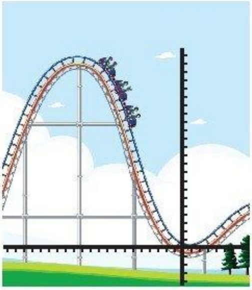
$\Rightarrow \quad f'(x) = \dfrac{2x}{4} - \dfrac{3}{2} = \dfrac{x}{2} - \dfrac{3}{2}$

$\Rightarrow \quad f'(-1) = \dfrac{-1}{2} - \dfrac{3}{2} = \dfrac{-4}{2} = -2$

$\Rightarrow \quad f'(-1) = -2$

2. Read the following passage and answer the following questions. [*CBSE 2023 (65/1/1)*]

The equation of the path traced by a roller-coaster is given by the polynomial $f(x) = a(x+9)(x+1)(x-3)$. If the roller-coaster crosses y-axis at a point $(0, -1)$, answer the following:



(*i*) Find the value of 'a'.

(*ii*) Find $f''(x)$ at $x = 1$.

Sol. (*i*) Given polynomial $f(x) = a(x+9)(x+1)(x-3)$.

Since polynomial $f(x)$ crosses y-axis at $(0, -1)$.

$\therefore \quad -1 = a(0+9)(0+1)(0-3)$

$\Rightarrow \quad -1 = a \times (-27) \quad \Rightarrow a = \dfrac{1}{27}$

(*ii*) We have,

$$f(x) = a(x+9)(x+1)(x-3)$$

$\Rightarrow \quad f(x) = \dfrac{1}{27}(x^3 + 7x^2 - 21x - 27)$

$\Rightarrow \quad f'(x) = \dfrac{1}{27}(3x^2 + 14x - 21)$

$\Rightarrow \quad f''(x) = \dfrac{1}{27}(6x + 14)$

$\Rightarrow \quad f''(x) = \dfrac{2}{27}(3x + 7)$

$\therefore \quad f''(1) = \dfrac{2}{27}(3+7) = \dfrac{20}{27}$

CONCEPTUAL QUESTIONS

1. **If the function f defined as $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & x\neq 3\\ k, & x=3\end{cases}$ is continuous at $x = 3$, find the value of k.**

[CBSE 2020 (65/5/1)]

Sol. $\lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}=6, \ \therefore k=6$ 1

[CBSE Marking Scheme 2020 (65/5/1)]

Detailed Solution:

$\because$ It is given that $f(x)$ is continuous at $x = 3$.

$\therefore \quad \lim\limits_{x\to 3} f(x) = f(3)$

$\Rightarrow \quad \lim\limits_{x\to 3}\dfrac{x^2-9}{x-3}=k \quad \Rightarrow \quad \lim\limits_{x\to 3}\dfrac{(x-3)(x+3)}{(x-3)}=k$

$\Rightarrow \quad \lim\limits_{x\to 3}(x+3)=k \quad \Rightarrow \quad 3+3=k \quad \Rightarrow \quad k=6$

2. **For what value of 'k' is the function $f(x)=\begin{cases}\dfrac{\sin 5x}{3x}+\cos x, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$ continuous at $x = 0$?**

[CBSE (F) 2017]

Sol. $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{h\to 0} f(0+h)$

$$=\lim_{h\to 0} f(h)=\lim_{h\to 0}\left(\frac{\sin 5h}{3h}+\cos h\right)$$

$$=\lim_{h\to 0}\frac{\sin 5h}{5h}\times\frac{5}{3}+\lim_{h\to 0}\cos h = 1\times\frac{5}{3}+1 \qquad [\because\ h\to 0\Rightarrow 5h\to 0]$$

$\Rightarrow \quad \lim\limits_{x\to 0^+} f(x)=\dfrac{8}{3}$

Also, $\quad f(0)=k$

Since, $f(x)$ is continuous at $x = 0$.

$\Rightarrow \quad \lim\limits_{x\to 0^+} f(x)=f(0) \quad \Rightarrow \quad \dfrac{8}{3}=k$

3. **Determine value of the constant 'k' so that the function $f(x)=\begin{cases}\dfrac{kx}{|x|}, & \text{if } x<0\\ 3, & \text{if } x\geq 0\end{cases}$ is continuous at $x = 0$.**

[CBSE Delhi 2017]

Sol. $\because \ f(x)$ is continuous at $x = 0$.

$\Rightarrow \quad \lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^-} f(x)=f(0)$

Now, $\quad \lim\limits_{x\to 0^-} f(x)=\lim\limits_{h\to 0} f(0-h)$

$$=\lim_{h\to 0} f(-h)=\lim_{h\to 0}\frac{k(-h)}{|-h|}=\lim_{h\to 0}\frac{-kh}{h}=-k$$

Also, $f(0) = 3$

$\because \quad \lim\limits_{x\to 0^-} f(x)=f(0) \quad \Rightarrow \quad -k=3 \quad \Rightarrow \quad k=-3$

4. Differentiate $\sec^2(x^2)$ with respect to x^2. [CBSE 2020 (65/2/1)]

Sol. Let $x^2 = u$, differentiating $\sec^2 u$ w.r.t u, putting $u = x^2$, we get

$2\sec^2 x^2 \tan x^2$ **1**

[CBSE Marking Scheme 2020 (65/2/1)]

5. If $y = f(x^2)$ and $f'(x) = e^{\sqrt{x}}$, then find $\frac{dy}{dx}$. [CBSE 2020 (65/2/1)]

Sol. $\frac{dy}{dx} = f'(x^2)2x$

$= 2xe^x$ **1**

[CBSE Marking Scheme 2020 (65/2/1)]

6. Differentiate $\sin^2(\sqrt{x})$ with respect to x. [CBSE 2020 (65/3/1)]

Sol. $\frac{\sin(2\sqrt{x})}{(2\sqrt{x})}$ or $\frac{\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}$ **1**

[CBSE Marking Scheme 2020 (65/3/1)]

7. If $y = 2\sqrt{\sec(e^{2x})}$ then find $\frac{dy}{dx}$. [CBSE 2019 (65/5/3)]

Sol. Given, $y = 2\sqrt{\sec(e^{2x})}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(2\sqrt{\sec(e^{2x})}\right) = 2\times\frac{1}{2\sqrt{\sec(e^{2x})}}\times\sec(e^{2x})\tan(e^{2x})\times 2e^{2x}$$

$$= 2\sqrt{\sec(e^{2x})}\tan(e^{2x}).e^{2x} = 2e^{2x}\sqrt{\sec(e^{2x})}\tan(e^{2x})$$

8. If $y = \text{cosec}(\cot\sqrt{x})$ then find $\frac{dy}{dx}$. [CBSE 2019 (65/4/2)]

Sol. Given, $y = \text{cosec}(\cot\sqrt{x})$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\text{cosec}(\cot\sqrt{x}))$$

$$= -\text{cosec}(\cot\sqrt{x})\cot(\cot\sqrt{x})\times\left(-\text{cosec}^2(\sqrt{x})\times\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{\text{cosec}(\cot\sqrt{x})\cot(\cot\sqrt{x})\times\text{cosec}^2(\sqrt{x})}{2\sqrt{x}}$$

9. Find the derivative of $\log_{10} x$ with respect to x. [NCERT Exemplar]

Sol. Let $y = \log_{10} x = \log_{10} e . \log_e x$

$$\therefore \frac{dy}{dx} = \log_{10}e\times\frac{1}{x} = \frac{\log_{10}e}{x} \qquad \left[\because \frac{d}{dx}\log_e x = \frac{1}{x}\right]$$

10. If $y = 5e^{7x} + 6e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$. [CBSE 2019 (65/5/1)]

Sol. Given, $y = 5e^{7x} + 6e^{-7x}$

$$\Rightarrow \frac{dy}{dx} = 35e^{7x} - 42e^{-7x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 245e^{7x} + 294e^{-7x} = 49(5e^{7x} + 6e^{-7x}) = 49y$$

4. Differentiate $\sec^2(x^2)$ with respect to x^2. [CBSE 2020 (65/2/1)]

Sol. Let $x^2 = u$, differentiating $\sec^2 u$ w.r.t u, putting $u = x^2$, we get

$2\sec^2 x^2 \tan x^2$ 1

[CBSE Marking Scheme 2020 (65/2/1)]

5. If $y = f(x^2)$ and $f'(x) = e^{\sqrt{x}}$, then find $\frac{dy}{dx}$. [CBSE 2020 (65/2/1)]

Sol.
$$\frac{dy}{dx} = f'(x^2)2x$$
$$= 2xe^x$$
1

[CBSE Marking Scheme 2020 (65/2/1)]

6. Differentiate $\sin^2(\sqrt{x})$ with respect to x. [CBSE 2020 (65/3/1)]

Sol. $\frac{\sin(2\sqrt{x})}{(2\sqrt{x})}$ or $\frac{\sin\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}$ 1

[CBSE Marking Scheme 2020 (65/3/1)]

7. If $y = 2\sqrt{\sec(e^{2x})}$ then find $\frac{dy}{dx}$. [CBSE 2019 (65/5/3)]

Sol. Given, $y = 2\sqrt{\sec(e^{2x})}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(2\sqrt{\sec(e^{2x})}\right) = 2\times\frac{1}{2\sqrt{\sec(e^{2x})}}\times\sec(e^{2x})\tan(e^{2x})\times 2e^{2x}$$

$$= 2\sqrt{\sec(e^{2x})}\tan(e^{2x}).e^{2x} = 2e^{2x}\sqrt{\sec(e^{2x})}\tan(e^{2x})$$

8. If $y = \text{cosec}(\cot\sqrt{x})$ then find $\frac{dy}{dx}$. [CBSE 2019 (65/4/2)]

Sol. Given, $y = \text{cosec}(\cot\sqrt{x})$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(\text{cosec}(\cot\sqrt{x}))$$

$$= -\text{cosec}(\cot\sqrt{x})\cot(\cot\sqrt{x})\times\left(-\text{cosec}^2(\sqrt{x})\times\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{\text{cosec}(\cot\sqrt{x})\cot(\cot\sqrt{x})\times\text{cosec}^2(\sqrt{x})}{2\sqrt{x}}$$

9. Find the derivative of $\log_{10} x$ with respect to x. [NCERT Exemplar]

Sol. Let $y = \log_{10} x = \log_{10} e . \log_e x$

$$\therefore \quad \frac{dy}{dx} = \log_{10}e\times\frac{1}{x} = \frac{\log_{10}e}{x} \qquad \left[\because \frac{d}{dx}\log_e x = \frac{1}{x}\right]$$

10. If $y = 5e^{7x} + 6e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$. [CBSE 2019 (65/5/1)]

Sol. Given, $y = 5e^{7x} + 6e^{-7x}$

$$\Rightarrow \quad \frac{dy}{dx} = 35e^{7x} - 42e^{-7x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 245e^{7x} + 294e^{-7x} = 49(5e^{7x} + 6e^{-7x}) = 49y$$

11. If $y = \log(\cos e^x)$ then find $\frac{dy}{dx}$. [CBSE 2019 (65/4/1)]

Sol. Given, $y = \log(\cos e^x)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(\log(\cos e^x)) = \frac{1}{\cos(e^x)} \times (-\sin(e^x)) \times e^x = \frac{-e^x \sin(e^x)}{\cos(e^x)} = -e^x \tan(e^x)$$

$$\therefore \quad \frac{dy}{dx} = -e^x \tan(e^x)$$

Very Short Answer Questions

1. Find the values(s) of 'λ', if the function $f(x) = \begin{cases} \frac{\sin^2 \lambda x}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$. [CBSE 2023 (65/5/1)]

Sol. $\because \ f(x) = \begin{cases} \frac{\sin^2 \lambda x}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

$$\therefore \quad \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin^2 \lambda x}{x^2} = \lim_{x \to 0} \frac{\sin^2 \lambda x}{\lambda^2 x^2} \times \lambda^2$$

$$= \lim_{x \to 0} \left(\frac{\sin(\lambda x)}{\lambda x}\right)^2 \times \lambda^2 = 1 \times \lambda^2 = \lambda^2$$

and $\quad f(0) = 1$

f is continuous at $x = 0$ if $\lim_{x \to 0} f(x) = f(0)$.

$$\Rightarrow \quad \lambda^2 = 1 \ \Rightarrow \ \lambda = \pm 1$$

2. If $f(x) = \begin{cases} ax + b & ; \ 0 < x \leq 1 \\ 2x^2 - x & ; \ 1 < x < 2 \end{cases}$ is a differentiable function in (0, 2) then find the values of a and b. [CBSE 2023 (65/2/1)]

Sol. Given function, $f(x) = \begin{cases} ax + b \ ; & 0 < x \leq 1 \\ 2x^2 - x \ ; & 1 < x < 2 \end{cases}$ is differentiable in (0, 2).

at $x = 1$

LHD $= \lim_{x \to 1^-} f'(x) = a$

RHD $= \lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} 4x - 1 = 4 \times 1 - 1 = 3$

Since $f(x)$ is differentiable in (0, 2).

$\therefore$ LHD = RHD at $x = 1$

$\Rightarrow \quad a = 3$

As f is differentiable in (0, 2) so is continuous.

At $x = 1$, $\quad$ LHL $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (ax + b) = a + b$

RHL $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x^2 - x) = 2 - 1 = 1$

$\Rightarrow \quad$ LHL = RHL

$\Rightarrow \quad a + b = 1 \quad \Rightarrow 3 + b = 1 \quad \Rightarrow \quad b = -2$

$\therefore \quad a = 3, b = -2$

3. If $y = \left(x + \sqrt{x^2 - 1}\right)^2$, then show that $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y^2$. *[CBSE 2023 (65/1/1)]*

Sol. We have,

$$y = \left(x + \sqrt{x^2 - 1}\right)^2$$

Differentiating w.r.t. x, we have

$$\frac{dy}{dx} = 2\left(x + \sqrt{x^2 - 1}\right) \times \left(1 + \frac{1}{2\sqrt{x^2 - 1}} \times 2x\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\left(x + \sqrt{x^2 - 1}\right) \times \left(x + \sqrt{x^2 - 1}\right)}{\sqrt{x^2 - 1}} = \frac{2\left(x + \sqrt{x^2 - 1}\right)^2}{\sqrt{x^2 - 1}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2y}{\sqrt{x^2 - 1}}$$

Squaring both sides, we have

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{4y^2}{(x^2 - 1)} \qquad \Rightarrow (x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y^2$$

4. If $(x^2 + y^2)^2 = xy$ then find $\frac{dy}{dx}$. *[CBSE 2023 (65/4/1)]*

Sol. Given,

$$(x^2 + y^2)^2 = xy$$

Differentiating w.r.t x, we get

$$2(x^2 + y^2) \times \left\{2x + 2y\frac{dy}{dx}\right\} = x\frac{dy}{dx} + y \cdot 1$$

$$\Rightarrow \quad 4x\,(x^2 + y^2) + 4y\,(x^2 + y^2)\,\frac{dy}{dx} = x\frac{dy}{dx} + y$$

$$\Rightarrow \quad \{4y(x^2 + y^2) - x\}\,\frac{dy}{dx} = y - 4x(x^2 + y^2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{y - 4x\left(x^2 + y^2\right)}{4y\left(x^2 + y^2\right) - x}$$

5. If $f(x) = \begin{cases} x^2, \text{ if } x \geq 1 \\ x\,, \text{ if } x < 1 \end{cases}$, then show that f is not differentiable at $x = 1$. *[CBSE 2023 (65/5/1)]*

Sol. $\because \; f(x) = \begin{cases} x^2, & \text{if } x \geq 1 \\ x\,, & \text{if } x < 1 \end{cases}$

At $x = 1$

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h) - 1^2}{-h} \qquad [\because 1 - h < 1]$$

$$= \lim_{h \to 0} \frac{1 - h - 1}{-h} = \lim_{h \to 0} \frac{-h}{-h} = \lim_{h \to 0} (1)$$

$$= 1$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \qquad [\because 1 + h > 1]$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h \to 0}(2 + h) = 2 + 0 = 2$$

As LHD ≠ RHD at $x = 1$

$\Rightarrow$ f is not differentiable at $x = 1$.

6. If $y = x^{\frac{1}{x}}$, then find $\frac{dy}{dx}$ at $x = 1$. [CBSE 2023 (65/3/2)]

Sol. Given, $y = x^{1/x}$

Taking log on both sides, we have

$$\log y = \frac{1}{x} \log x = \frac{\log x}{x}$$

Differentiate w.r.t x, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{x \times \frac{1}{x} - \log x \times 1}{x^2} = \frac{1 - \log x}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(1 - \log x)}{x^2} = \frac{x^{1/x}(1 - \log x)}{x^2}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } x=1} = \frac{1(1 - \log 1)}{1} = \frac{1-0}{1} = 1$$

7. If $x = a \sin 2t$, $y = a(\cos 2t + \log \tan t)$, then find $\frac{dy}{dx}$. [CBSE 2023 (65/3/2)]

Sol. Given $x = a \sin 2t \Rightarrow \frac{dx}{dt} = a \cos 2t \times 2$

$$\Rightarrow \frac{dx}{dt} = 2a \cos 2t$$

and, $y = a(\cos 2t + \log \tan t)$

$$\Rightarrow \quad \frac{dy}{dt} = a\left(-2\sin 2t + \frac{1}{\tan t} \times \sec^2 t\right) = a\left(-2\sin 2t + \frac{2}{\sin 2t}\right)$$

$$\Rightarrow \quad \frac{dy}{dt} = 2a\left(\frac{1 - \sin^2 2t}{\sin 2t}\right) = \frac{2a\cos^2 2t}{\sin 2t}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{2a\cos^2 2t}{\sin 2t}}{2a\cos 2t} = \frac{\cos 2t}{\sin 2t} = \cot 2t$$

8. If $x = at^2, y = 2at$, then find $\frac{d^2y}{dx^2}$. [CBSE 2020 (65/1/1)]

Sol. Given, $x = at^2 \Rightarrow \frac{dx}{dt} = 2at$

Also, $y = 2at \Rightarrow \frac{dy}{dt} = 2a$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right) = -\frac{1}{t^2}\times\frac{dt}{dx}$$

$$= -\frac{1}{t^2}\times\frac{1}{2at} = -\frac{1}{2at^3}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{2at^3}$$

9. If $f(x) = \sqrt{\tan\sqrt{x}}$, then find $f'\left(\frac{\pi^2}{16}\right)$. *[CBSE 2020 (65/1/3)]*

Sol. Given, $x = at^2 \Rightarrow \frac{dx}{dt} = 2at$

Also, $y = 2at \Rightarrow \frac{dy}{dt} = 2a$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right) = -\frac{1}{t^2}\times\frac{dt}{dx}$$

$$= -\frac{1}{t^2}\times\frac{1}{2at} = -\frac{1}{2at^3}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{2at^3}$$

10. If $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$, then show that $\frac{dy}{dx} = \frac{y^2\tan x}{y\log\cos x - 1}$. *[NCERT Exemplar]*

Sol. We have, $y = (\cos x)^{(\cos x)^{(\cos x)^{\cdots\infty}}}$

$\Rightarrow \quad y = (\cos x)^y$

$\therefore \quad \log y = \log(\cos x)^y \quad \Rightarrow \quad \log y = y\log\cos x$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{d}{dx}\log\cos x + \log\cos x\cdot\frac{dy}{dx} \quad \Rightarrow \quad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{\cos x}\cdot\frac{d}{dx}\cos x + \log\cos x\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{1}{y} - \log\cos x\right] = \frac{-y\sin x}{\cos x} = -y\tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y^2\tan x}{(1 - y\log\cos x)} = \frac{y^2\tan x}{y\log\cos x - 1}$$

11. If $x = a\cos\theta$; $y = b\sin\theta$, then find $\frac{d^2y}{dx^2}$. *[CBSE 2020 (65/5/1)]*

Sol. $\frac{dx}{d\theta} = -a\sin\theta,\ \frac{dy}{d\theta} = b\cos\theta \Rightarrow \frac{dy}{dx} = -\frac{b}{a}\cot\theta$ **1**

$\frac{d^2y}{dx^2} = \frac{b}{a}\operatorname{cosec}^2\theta\left(\frac{-1}{a\sin\theta}\right) = \frac{-b}{a^2}\operatorname{cosec}^3\theta$ **1**

[CBSE Marking Scheme 2020 (65/5/1)]

Detailed Solution:

We have,

$$x = a\cos\theta \Rightarrow \frac{dx}{d\theta} = -a\sin\theta$$

and $$y = b\sin\theta \Rightarrow \frac{dy}{d\theta} = b\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

Again differentiating w.r.t x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{b}{a}\frac{d}{dx}(\cot\theta) = -\frac{b}{a}\times(-\operatorname{cosec}^2\theta)\frac{d\theta}{dx}$$

$$= -\frac{b}{a}\times(-\operatorname{cosec}^2\theta)\times\left(-\frac{1}{a\sin\theta}\right)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{b}{a^2}\operatorname{cosec}^3\theta$$

12. If $y = \sqrt{ax+b}$, prove that $y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0$. *[CBSE 2023 (65/2/1)]*

Sol. Given, $y = \sqrt{ax+b} \Rightarrow y^2 = (ax+b)$

Differentiate w.r.t x, we have

$$2yy' = a \quad \Rightarrow yy' = \frac{a}{2}$$

Again differentiate w.r.t x, we have

$$yy'' + y'.y' = 0 \quad \Rightarrow yy'' + (y')^2 = 0$$

$$\Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

Short Answer Questions

Continuity and Differentiability

1. Find the values of p and q, for which *[CBSE Delhi 2016]*

$$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi - 2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$$ **is continuous at $x = \frac{\pi}{2}$.**

Sol. We have, $$f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi - 2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$$ is continuous at $x = \frac{\pi}{2}$.

Now, $\lim_{x \to \frac{\pi^+}{2}} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2}+h\right)$ $\left[\text{Let } x = \frac{\pi}{2}+h, x \to \frac{\pi^+}{2} \Rightarrow h \to 0\right]$

$$= \lim_{h \to 0} \frac{q\left\{1-\sin\left(\frac{\pi}{2}+h\right)\right\}}{\left\{\pi - 2\left(\frac{\pi}{2}+h\right)\right\}^2} = \lim_{h \to 0} \frac{q\{1-\cos h\}}{\{\pi-\pi-2h\}^2} = \lim_{h \to 0} \frac{q(1-\cos h)}{4h^2}$$

$$= \lim_{h \to 0} \frac{q.2\sin^2\frac{h}{2}}{4h^2} = \lim_{h \to 0} \frac{q.\sin^2\frac{h}{2}}{2h^2} = q.\lim_{h \to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 \times \frac{1}{8} = \frac{q}{8}$$

Again $\lim_{x \to \frac{\pi^-}{2}} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2}-h\right)$ $\left[\text{Let } x = \frac{\pi}{2}-h, x \to \frac{\pi^-}{2} \Rightarrow h \to 0\right]$

$$= \lim_{h \to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)} = \lim_{h \to 0} \frac{1-\cos^3 h}{3\sin^2 h} = \lim_{h \to 0} \frac{(1-\cos h)(1+\cos h+\cos^2 h)}{3\sin^2 h}$$

$$= \lim_{h \to 0} \frac{2\sin^2\frac{h}{2}}{3\sin^2 h} \times \lim_{h \to 0}(1+\cos h+\cos^2 h)$$

$$= \lim_{h \to 0} \frac{2\sin^2\frac{h}{2}}{3\sin^2 h} \times (1+1+1) = \lim_{h \to 0}\left(\frac{2\sin^2\frac{h}{2}}{\sin^2 h}\right)$$

Dividing N^r and D^r by h^2, we get

$$= \lim_{h \to 0} \frac{2.\frac{\sin^2\frac{h}{2}}{h^2}}{\frac{\sin^2 h}{h^2}} = \lim_{h \to 0} \frac{2.\frac{\sin^2\frac{h}{2}}{\frac{h^2}{4}\times 4}}{\frac{\sin^2 h}{h^2}} = \frac{1}{2}\frac{\left(\lim_{\frac{h}{2} \to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2}{\left(\lim_{h \to 0}\frac{\sin h}{h}\right)^2} = \frac{1}{2}$$

Also $f\left(\frac{\pi}{2}\right) = p$

$\because$ $f(x)$ is continuous at $x = \frac{\pi}{2}$. $\Rightarrow$ $\lim_{h \to \frac{\pi^+}{2}} f(x) = \lim_{h \to \frac{\pi^-}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

$\Rightarrow$ $\frac{q}{8} = \frac{1}{2} = p$ $\Rightarrow$ $p = \frac{1}{2}$ and $q = 4$

2. **Show that the function $f(x) = 2x - |x|$ is continuous but not differentiable at $x = 0$.** [*CBSE (F) 2013*]

Sol. Here $f(x) = 2x - |x|$

For continuity at $x = 0$

$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$= \lim_{h \to 0}\{2h - |h|\} = \lim_{h \to 0}(2h-h)$

$= \lim_{h \to 0} h$

$= 0$ ...(*i*)

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$= \lim_{h \to 0}\{2(-h) - |-h|\} = \lim_{h \to 0}\{-2h-h\}$

$= \lim_{h \to 0}(-3h)$

$= 0$ ...(*ii*)

Now, $\lim_{x \to \frac{\pi^+}{2}} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2}+h\right)$ $\left[\text{Let } x = \frac{\pi}{2}+h, x \to \frac{\pi^+}{2} \Rightarrow h \to 0\right]$

$$= \lim_{h \to 0} \frac{q\left\{1-\sin\left(\frac{\pi}{2}+h\right)\right\}}{\left\{\pi-2\left(\frac{\pi}{2}+h\right)\right\}^2} = \lim_{h \to 0} \frac{q\{1-\cos h\}}{\{\pi-\pi-2h\}^2} = \lim_{h \to 0} \frac{q(1-\cos h)}{4h^2}$$

$$= \lim_{h \to 0} \frac{q.2\sin^2\frac{h}{2}}{4h^2} = \lim_{h \to 0} \frac{q.\sin^2\frac{h}{2}}{2h^2} = q.\lim_{h \to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2 \times \frac{1}{8} = \frac{q}{8}$$

Again $\lim_{x \to \frac{\pi^-}{2}} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2}-h\right)$ $\left[\text{Let } x = \frac{\pi}{2}-h, x \to \frac{\pi^-}{2} \Rightarrow h \to 0\right]$

$$= \lim_{h \to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)} = \lim_{h \to 0} \frac{1-\cos^3 h}{3\sin^2 h} = \lim_{h \to 0} \frac{(1-\cos h)(1+\cos h+\cos^2 h)}{3\sin^2 h}$$

$$= \lim_{h \to 0} \frac{2\sin^2\frac{h}{2}}{3\sin^2 h} \times \lim_{h \to 0}(1+\cos h+\cos^2 h)$$

$$= \lim_{h \to 0} \frac{2\sin^2\frac{h}{2}}{3\sin^2 h} \times (1+1+1) = \lim_{h \to 0}\left(\frac{2\sin^2\frac{h}{2}}{\sin^2 h}\right)$$

Dividing N^r and D^r by h^2, we get

$$= \lim_{h \to 0} \frac{2.\frac{\sin^2\frac{h}{2}}{h^2}}{\frac{\sin^2 h}{h^2}} = \lim_{h \to 0} \frac{2.\frac{\sin^2\frac{h}{2}}{\frac{h^2}{4}\times 4}}{\frac{\sin^2 h}{h^2}} = \frac{1}{2}\frac{\left(\lim_{\frac{h}{2} \to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2}{\left(\lim_{h \to 0}\frac{\sin h}{h}\right)^2} = \frac{1}{2}$$

Also $f\left(\frac{\pi}{2}\right) = p$

$\because$ $f(x)$ is continuous at $x = \frac{\pi}{2}$. $\Rightarrow$ $\lim_{h \to \frac{\pi^+}{2}} f(x) = \lim_{h \to \frac{\pi^-}{2}} f(x) = f\left(\frac{\pi}{2}\right)$

$\Rightarrow$ $\frac{q}{8} = \frac{1}{2} = p$ $\Rightarrow$ $p = \frac{1}{2}$ and $q = 4$

2. **Show that the function $f(x) = 2x - |x|$ is continuous but not differentiable at $x = 0$.** [*CBSE (F) 2013*]

Sol. Here $f(x) = 2x - |x|$

For continuity at $x = 0$

$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$= \lim_{h \to 0}\{2h - |h|\} = \lim_{h \to 0}(2h-h)$

$= \lim_{h \to 0} h$

$= 0$ ...(*i*)

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$= \lim_{h \to 0}\{2(-h) - |-h|\} = \lim_{h \to 0}\{-2h-h\}$

$= \lim_{h \to 0}(-3h)$

$= 0$ ...(*ii*)

Also, $f(0) = 2 \times 0 - |0| = 0$ ...(*iii*)

(*i*), (*ii*) and (*iii*) $\Rightarrow \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^-} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

For differentiability at $x = 0$

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(2(-h) - |-h|) - \{2 \times 0 - |0|\}}{-h} = \lim_{h \to 0} \frac{-2h - h - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-3h}{-h} = \lim_{h \to 0} 3 = 3$$

$$\text{LHD} = 3 \qquad \text{...}(iv)$$

$$\text{Again RHD} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{2h - |h| - 2 \times 0 - |0|}{h} = \lim_{h \to 0} \frac{2h - h}{h} = \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1 = 1$$

$$\text{RHD} = 1 \qquad \text{...}(v)$$

From (*iv*) and (*v*), we get

LHD $\neq$ RHD *i.e.*, function $f(x) = 2x - |x|$ is not differentiable at $x = 0$.

Hence, $f(x)$ is continuous but not differentiable at $x = 0$.

3. Find the value of '*k*' for which the function *f* defined as

$$f(x) = \begin{cases} k \sin \frac{\pi}{2}(x+1), & x \leq 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$ **is continuous at $x = 0$.** [*CBSE Delhi 2011; (South) 2016*]

Sol.

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$$f(x) = \begin{cases} k\sin\left(\frac{\pi}{2}(x+1)\right) & x \le 0 \\ \frac{\tan x - \sin x}{x^3} & x > 0 \end{cases}$$

$f(x)$ is continuous at $x = 0$

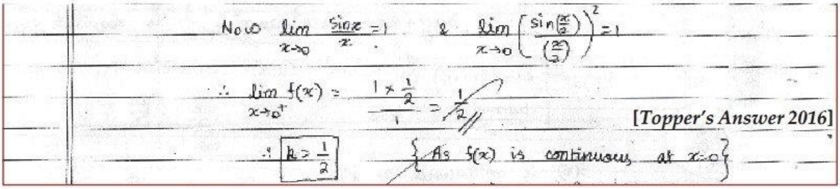
$\therefore \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0)$

Now $\lim\limits_{x \to 0^-} f(x) = f(0) = \lim\limits_{x \to 0^-} k\sin\left(\frac{\pi}{2}(x+1)\right) = k\sin\frac{\pi}{2} = k$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \frac{\tan x - \sin x}{x^3} = \lim\limits_{x \to 0^+} \frac{\sin x}{x} \left(\frac{\sec x - 1}{x^2}\right)$

$= \lim\limits_{x \to 0^+} \frac{\sin x}{x} \cdot \lim\limits_{x \to 0^+} \frac{1 - \cos x}{\cos x \times x^2}$

$= \lim\limits_{x \to 0^+} \frac{\sin x}{x} \times \dfrac{\lim\limits_{x \to 0^+} \dfrac{2\sin^2 \frac{x}{2}}{4 \times \left(\frac{x}{2}\right)^2}}{\lim\limits_{x \to 0^+} \cos x}$


Now $\lim_{x\to 0}\frac{\sin x}{x}=1$ & $\lim_{x\to 0}\left(\frac{\sin(\frac{x}{2})}{(\frac{x}{2})}\right)^2=1$

$\therefore \lim_{x\to 0^+} f(x) = \frac{1\times\frac{1}{2}}{1} = \frac{1}{2}$

$\therefore$ $k=\frac{1}{2}$ {As f(x) is continuous at x=0}

[Topper's Answer 2016]

4. If $f(x)=\begin{cases}\dfrac{\sin(a+1)x+2\sin x}{x} & , x<0\\ 2 & , x=0\\ \dfrac{\sqrt{1+bx}-1}{x} & , x>0\end{cases}$ is continuous at $x=0$, then find the values of a and b.

[CBSE (North) 2016]

Sol. We have, $f(x)=\begin{cases}\dfrac{\sin(a+1)x+2\sin x}{x} & , x<0\\ 2 & , x=0\\ \dfrac{\sqrt{1+bx}-1}{x} & , x>0\end{cases}$ is continuous at $x=0$.

Since, $f(x)$ is continuous at $x=0$.

$\Rightarrow \quad \lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} f(x)=f(0) \qquad \ldots (i)$

Now, $\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)$ [Let $x=0+h$, h is +ve small quantity $x\to 0^+\Rightarrow h\to 0$]

$$=\lim_{h\to 0} f(h)=\lim_{h\to 0}\frac{\sqrt{1+bh}-1}{h}=\lim_{h\to 0}\frac{\sqrt{1+bh}-1}{h}\times\frac{\sqrt{1+bh}+1}{\sqrt{1+bh}+1}$$

$$=\lim_{h\to 0}\frac{1+bh-1}{h(\sqrt{1+bh}+1)}=\lim_{h\to 0}\frac{bh}{h(\sqrt{1+bh}+1)}=\lim_{h\to 0}\frac{b}{\sqrt{1+bh}+1}=\frac{b}{2}$$

Again $\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)$ [Let $x=0-h$, h is +ve small quantity $x\to 0^-\Rightarrow h\to 0$]

$$=\lim_{h\to 0} f(-h)=\lim_{h\to 0}\frac{\sin(a+1)(-h)+2\sin(-h)}{-h}$$

$$=\lim_{h\to 0}\left[\frac{-\sin(a+1)h-2\sin h}{-h}\right]=\lim_{h\to 0}\left[\frac{\sin(a+1)h}{h}+\frac{2\sin h}{h}\right]$$

$$=\lim_{h\to 0}\frac{\sin(a+1)h}{h}+2\lim_{h\to 0}\frac{\sin h}{h}=\lim_{h\to 0}\frac{\sin(a+1)h}{(a+1)h}\times(a+1)+2\lim_{h\to 0}\frac{\sin h}{h}$$

$$=1\times(a+1)+2 \qquad =a+3$$

Also $\quad f(0)=2$

Now from (*i*) $\dfrac{b}{2}=a+3=2 \quad\Rightarrow\quad b=4,\ a=-1$

5. Show that the function $f(x)=|x-3|, x\in\mathbb{R}$, is continuous but not differentiable at $x=3$.

[CBSE Delhi 2013]

Sol. Here, $f(x)=|x-3| \quad\Rightarrow\quad f(x)=\begin{cases}-(x-3) & , x<3\\ 0 & , x=3\\ (x-3) & , x>3\end{cases}$

For Continuity:

Now, $\lim_{x\to 3^+} f(x)=\lim_{h\to 0} f(3+h)$ [Let $x=3+h$ and $x\to 3^+ \Rightarrow h\to 0$]

$$= \lim_{h \to 0} (3 + h - 3) = \lim_{h \to 0} h = 0$$

$$\lim_{x \to 3^+} f(x) = 0 \qquad ...(i)$$

$$\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3 - h) \qquad [\text{Let } x = 3 - h \text{ and } x \to 3^- \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} -(3 - h - 3) = \lim_{h \to 0} h = 0$$

$$\lim_{x \to 3^+} f(x) = 0 \qquad ...(ii)$$

Also, $f(3) = 0$ ...(*iii*)

From equations (*i*), (*ii*) and (*iii*), we get

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) = f(3)$$

Hence, $f(x)$ is continuous at $x = 3$.

For Differentiability:

$$\text{RHD} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3 + h - 3) - 0}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1 \qquad ...(iv)$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(3-h) - f(3)}{-h} = \lim_{h \to 0} \frac{-(3 - h - 3) - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} (-1) = -1 \qquad ...(v)$$

From equations (*iv*) and (*v*), we get

RHD $\neq$ LHD at $x = 3$.

Hence, $f(x)$ is not differentiable at $x = 3$.

Therefore, $f(x) = |x - 3|, x \in \mathbb{R}$ is continuous but not differentiable at $x = 3$.

6. Discuss the continuity and differentiability of the function

$f(x) = |x| + |x - 1|$ in the interval $(-1, 2)$. [*CBSE Ajmer 2015*]

Sol. Given function is $f(x) = |x| + |x - 1|$

Function is also written as

$$f(x) = \begin{cases} -x-(x-1), & \text{if } -1 < x < 0 \\ 1, & \text{if } 0 \le x < 1 \\ x + (x-1), & \text{if } x \ge 1 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} -2x + 1, & \text{if } x < 0 \\ 1, & \text{if } 0 \le x < 1 \\ 2x - 1, & \text{if } x \ge 1 \end{cases}$$

Obviously, in given function we need to discuss the continuity and differentiability of the function $f(x)$ at $x = 0$ or 1 only.

For continuity at $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) \qquad [\text{Let } x = 0 + h \text{ and } x \to 0^+ \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} 1 \qquad [\because h \text{ is very small positive quantity}]$$

$$= 1 \qquad ...(i)$$

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) \qquad [\text{Let } x = 0 - h \text{ and } x \to 0^- \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \{-2(-h) + 1\} = \lim_{h \to 0} (2h + 1)$$

$$\lim_{x \to 0^-} f(x) = 1 \qquad ...(ii)$$

Also, $f(0) = 1$ ...(*iii*)

(*i*), (*ii*) and (*iii*) $\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

For differentiability at $x = 0$

$$\text{RHD} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \quad [\because\ h \text{ is very small positive quantity}]$$

$$= \lim_{h \to 0} \frac{1-1}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0 \quad [\because\ |h| = h,\ |0| = 0]$$

$$\text{RHD} = 0 \quad \ldots(iv)$$

$$\text{LHD} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-2(-h) + 1 - 1}{-h} = \lim_{h \to 0} \frac{2h}{-h} = \lim_{h \to 0} (-2) = -2$$

$$\text{LHD} = -2 \quad \ldots(v)$$

(iv) and (v) $\Rightarrow$ RHD $\neq$ LHD at $x = 0$.

Hence, $f(x)$ is not differentiable at $x = 0$ but continuous at $x = 0$.

Similarly, we can prove $f(x)$ is not differentiable at $x = 1$ but continuous at $x = 1$. (Do yourself)

7. Find 'a' and 'b', if the function given by $f(x) = \begin{cases} ax^2 + b, \text{ if } x < 1 \\ 2x + 1, \text{ if } x \geq 1 \end{cases}$ is differentiable at $x = 1$.

[*CBSE Sample Paper 2018*]

Sol. Since, f is differentiable at 1. $\Rightarrow$ f is also continuous at 1.

Now $\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$ [Here h is +ve and very small quantity]

$$= \lim_{h \to 0} 2(1+h) + 1 = 2 + 1 = 3$$

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \{a(1-h)^2 + b\} = a + b$$

Since $f(x)$ is continuous at $x = 1$.

$\Rightarrow$ $a + b = 3$ $\ldots(i)$

Again, since f is differentiable.

$\Rightarrow$ LHD (at $x = 1$) = RHD (at $x = 1$) $\Rightarrow$ $\lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$

$\Rightarrow$ $\lim_{h \to 0} \frac{a(1-h)^2 + b - 3}{-h} = \lim_{h \to 0} \frac{2(1+h) + 1 - 3}{h}$ $\Rightarrow$ $\lim_{h \to 0} \frac{a - 2ah + ah^2 + b - 3}{-h} = \lim_{h \to 0} \frac{2 + 2h + 1 - 3}{h}$

$\Rightarrow$ $\lim_{h \to 0} \frac{-2ah + ah^2 + (a+b) - 3}{-h} = \lim_{h \to 0} \frac{2h}{h}$ $\Rightarrow$ $\lim_{h \to 0} \frac{-2ah + ah^2 + 3 - 3}{-h} = 2$

$\Rightarrow$ $\lim_{h \to 0} \frac{ah(2-h)}{h} = 2$ $\Rightarrow$ $2a = 2$ $\Rightarrow$ $a = 1$ $\Rightarrow$ $b = 2$ [From equation (i)]

Derivatives

1. If $\tan^{-1}\left(\frac{y}{x}\right) = \log\sqrt{x^2 + y^2}$, prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$. **[*CBSE 2020 (65/2/1)*]**

Sol. Differentiating both sides w.r.t. x, we get

$$\frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{2x + 2y\frac{dy}{dx}}{2\sqrt{x^2 + y^2}}$$

1½ + 1½

Simplifying we get $x\frac{dy}{dx} - y = x + y\frac{dy}{dx}$ ½

getting $\frac{dy}{dx} = \frac{x+y}{x-y}$ ½

[*CBSE Marking Scheme 2020 (65/2/1)*]

Detailed Solution:

Given, $\tan^{-1}\left(\frac{y}{x}\right) = \log\sqrt{x^2+y^2}$

$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2+y^2)$

Differentiating w.r.t x, we have

$$\Rightarrow \quad \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left\{\frac{x\frac{dy}{dx} - y \times 1}{x^2}\right\} = \frac{1}{2} \times \frac{1}{x^2+y^2} \times \left(2x + 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{x^2}{x^2+y^2} \times \frac{\left(x\frac{dy}{dx} - y\right)}{x^2} = \frac{1}{x^2+y^2}\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = x + y\frac{dy}{dx}$$

$$\Rightarrow \quad (x-y)\frac{dy}{dx} = x + y \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{x+y}{x-y}$$

2. Differentiate $\tan^{-1}\frac{3x-x^3}{1-3x^2}, |x| < \frac{1}{\sqrt{3}}$ w.r.t. $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$. [*CBSE 2019 (65/5/1)*]

Sol. Let $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, Put $x = \tan\theta$ ½

$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right) \Rightarrow y = \tan^{-1}(\tan 3\theta) = 3\theta$

$y = 3\tan^{-1}x \Rightarrow \frac{dy}{dx} = \frac{3}{1+x^2}$ ...(*i*) 1

Let $z = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, put $x = \sin\phi$ ½

$z = \tan^{-1}\left(\frac{\sin\phi}{\sqrt{1-\sin^2\phi}}\right) \Rightarrow z = \tan^{-1}(\tan\phi) = \phi$

$z = \phi = \sin^{-1}x \Rightarrow \frac{dz}{dx} = \frac{1}{\sqrt{1-x^2}}$ ...(*ii*) 1

Using (*i*) & (*ii*), $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{3\sqrt{1-x^2}}{1+x^2}$ 1

[*CBSE Marking Scheme 2019 (65/5/1)*]

3. If $y^x = e^{y-x}$, then prove that $\dfrac{dy}{dx} = \dfrac{(1+\log y)^2}{\log y}$. *[CBSE (AI) 2013]*

Sol. Given, $y^x = e^{y-x}$

Taking logarithm both sides, we get $\log y^x = \log e^{y-x}$

$\left[\because \begin{array}{l} (i)\ \log_e mn = \log_e m + \log_e n \\ (ii)\ \log_e \frac{m}{n} = \log_e m - \log_e n \\ (iii)\ \log_e m^n = n \log_e m \\ (iv)\ \log e = 1 \end{array}\right]$

$\Rightarrow \quad x.\log y = (y-x)\,.\,\log e \qquad \Rightarrow \qquad x\,.\,\log y = (y-x)$

$\Rightarrow \quad x\,(1+\log y) = y \qquad \Rightarrow \qquad x = \dfrac{y}{1+\log y}$

Differentiating both sides with respect to y, we get

$$\frac{dx}{dy} = \frac{(1+\log y).1 - y.\left(0+\frac{1}{y}\right)}{(1+\log y)^2} = \frac{1+\log y - 1}{(1+\log y)^2} = \frac{\log y}{(1+\log y)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$$

4. If $(\cos x)^y = (\cos y)^x$, then find $\dfrac{dy}{dx}$. *[CBSE Delhi 2012]*

Sol. Given, $(\cos x)^y = (\cos y)^x$

Taking logarithm both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$$\Rightarrow \quad y\,.\log(\cos x) = x\,.\log(\cos y) \qquad [\because\ \log m^n = n\log m]$$

Differentiating both sides with respect to x, we get

$$y.\frac{1}{\cos x}(-\sin x) + \log(\cos x).\frac{dy}{dx} = x.\frac{1}{\cos y}.(-\sin y).\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \quad -\frac{y\sin x}{\cos x} + \log(\cos x).\frac{dy}{dx} = -\frac{x\sin y}{\cos y}.\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow \quad \log(\cos x).\frac{dy}{dx} + \frac{x\sin y}{\cos y}.\frac{dy}{dx} = \log(\cos y) + \frac{y\sin x}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\log(\cos x) + \frac{x\sin y}{\cos y}\right] = \log(\cos y) + \frac{y\sin x}{\cos x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log(\cos y) + \frac{y\sin x}{\cos x}}{\log(\cos x) + \frac{x\sin y}{\cos y}} = \frac{\log(\cos y) + y\tan x}{\log(\cos x) + x\tan y}$$

5. Find the value of $\dfrac{dy}{dx}$ at $\theta = \dfrac{\pi}{4}$, if $x = ae^\theta(\sin\theta - \cos\theta)$ and $y = ae^\theta(\sin\theta + \cos\theta)$. *[CBSE (AI) 2008, 2014]*

Sol. Given, $x = ae^\theta(\sin\theta - \cos\theta)$ and $y = ae^\theta(\sin\theta + \cos\theta)$

Taking $x = ae^\theta(\sin\theta - \cos\theta)$

Differentiating with respect to θ, we get

$$\frac{dx}{d\theta} = ae^\theta(\cos\theta + \sin\theta) + a(\sin\theta - \cos\theta).e^\theta = ae^\theta(\cos\theta + \sin\theta + \sin\theta - \cos\theta)$$

$$= 2\,ae^\theta\sin\theta \qquad \ldots (i)$$

Again, $y = ae^\theta(\sin\theta + \cos\theta)$

Differentiating with respect to θ, we get

$$\frac{dy}{d\theta} = ae^{\theta}(\cos\theta - \sin\theta) + a(\sin\theta + \cos\theta).e^{\theta} = ae^{\theta}(\cos\theta - \sin\theta + \sin\theta + \cos\theta)$$

$$= 2\,ae^{\theta}.\cos\theta \qquad \ldots (ii)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2ae^{\theta}.\cos\theta}{2ae^{\theta}.\sin\theta} \qquad \text{[From } (i) \text{ and } (ii)\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \cot\theta \quad \Rightarrow \quad \left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

6. Differentiate the following with respect to x: $(\sin x)^x + (\cos x)^{\sin x}$ *[CBSE (F) 2013]*

Sol. Let $u = (\sin x)^x$ and $v = (\cos x)^{\sin x}$

∴ Given differential equation becomes $y = u + v$.

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \ldots(i)$$

Now, $u = (\sin x)^x$

Taking log on both sides, we get

$$\log u = x \log \sin x$$

Differentiating with respect to x, we get

$$\frac{1}{u}.\frac{du}{dx} = x\frac{1}{\sin x}.\cos x + \log \sin x \quad \Rightarrow \quad \frac{du}{dx} = u(x\cot x + \log \sin x)$$

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^x\{x\cot x + \log \sin x\} \qquad \ldots(ii)$$

Again $v = (\cos x)^{\sin x}$

Taking log on both sides, we get

$$\log v = \sin x \,.\, \log \cos x$$

Differentiating both sides with respect to x, we get

$$\frac{1}{v}.\frac{dv}{dx} = \sin x.\frac{1}{\cos x}(-\sin x) + \log(\cos x).\cos x$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left\{-\frac{\sin^2 x}{\cos x} + \cos x.\log\cos x\right\} = (\cos x)^{\sin x}\left\{\cos x.\log(\cos x) - \frac{\sin^2 x}{\cos x}\right\}$$

$$= (\cos x)^{1+\sin x}\{\log(\cos x) - \tan^2 x\} \qquad \ldots(iii)$$

From (*i*), (*ii*) and (*iii*), we get

$$\frac{dy}{dx} = (\sin x)^x\{x\cot x + \log \sin x\} + (\cos x)^{1+\sin x}\{\log(\cos x) - \tan^2 x\}$$

7. If $x = a\sin 2t\,(1 + \cos 2t)$ and $y = b\cos 2t\,(1 - \cos 2t)$, then show that $\left.\frac{dy}{dx}\right._{\text{at } t=\frac{\pi}{4}} = \frac{b}{a}$.

[CBSE (AI) 2014; Panchkula 2015; (Central) 2016]

Sol.

$x = a\sin 2t\,(1+\cos 2t)$

$\frac{dx}{dt} = a[(\sin 2t)(-\sin 2t)(2) + (1+\cos 2t)\cos 2t \times 2]$

$\frac{dx}{dt} = 2a[\cos^2 2t + \cos 2t - \sin^2 2t]$

$\frac{dx}{dt} = 2a[\cos 4t + \cos 2t]$ $\quad [\cos^2 x - \sin^2 x = \cos 2x]$

$y = b\cos 2t(1 - \cos 2t)$

$y = b\cos 2t - b\cos^2 2t$

$\frac{dy}{dt} = -b\sin 2t \times 2 + b \times 2\cos 2t \sin 2t \times 2$

$\frac{dy}{dt} = 2b[-\sin 2t + \sin 4t]$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$\frac{dy}{dx} = \frac{2b}{2a}\left[\frac{\sin 4t - \sin 2t}{\cos 2t + \cos 4t}\right]$

at $t = \frac{\pi}{4}$

$\frac{dy}{dx} = \frac{b}{a}\left[\frac{\sin \pi - \sin\frac{\pi}{2}}{\cos\frac{\pi}{2} + \cos \pi}\right]$

$= \frac{b}{a}\left[\frac{0 - 1}{0 - 1}\right]$

$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{b}{a}$

[Topper's Answer 2016]

8. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$. *[CBSE (F) 2009, 2019(65/5/3)]*

Sol. $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, put $x = \sin\theta$, $y = \sin\phi$ **1**

$\Rightarrow \sqrt{1-\sin^2\theta} + \sqrt{1-\sin^2\phi} = a(\sin\theta - \sin\phi)$

$\Rightarrow \cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$\Rightarrow 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) = 2a\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$ **1**

$\Rightarrow \tan\left(\frac{\theta-\phi}{2}\right) = \frac{1}{a}$

$\Rightarrow \frac{\theta-\phi}{2} = \tan^{-1}\left(\frac{1}{a}\right) \quad \Rightarrow \sin^{-1}x - \sin^{-1}y = 2\tan^{-1}\left(\frac{1}{a}\right)$ **1**

Differentiating both sides w.r.t x, we get

$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$ **½**

$\Rightarrow \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ or $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ **½**

[CBSE Marking Scheme 2019 (65/5/3)]

9. If $x = \cos t(3 - 2\cos^2 t)$ and $y = \sin t(3 - 2\sin^2 t)$, then find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. [CBSE (AI) 2014]

Sol. Given, $x = \cos t(3 - 2\cos^2 t)$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = \cos t\{0 + 4\cos t.\sin t\} + (3 - 2\cos^2 t).(-\sin t)$$
$$= 4\sin t.\cos^2 t - 3\sin t + 2\cos^2 t.\sin t$$
$$= 6\sin t\cos^2 t - 3\sin t = 3\sin t(2\cos^2 t - 1) = 3\sin t.\cos 2t$$

Again, $\because y = \sin t(3 - 2\sin^2 t)$

Differentiating both sides with respect to t, we get

$$\frac{dy}{dt} = \sin t.\{0 - 4\sin t\cos t\} + (3 - 2\sin^2 t).\cos t = -4\sin^2 t.\cos t + 3\cos t - 2\sin^2 t.\cos t$$
$$= 3\cos t - 6\sin^2 t.\cos t = 3\cos t(1 - 2\sin^2 t) = 3\cos t.\cos 2t$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t.\cos 2t}{3\sin t.\cos 2t} \Rightarrow \frac{dy}{dx} = \cot t$$

$$\therefore \left.\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \cot\frac{\pi}{4} = 1$$

10. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. [CBSE Delhi 2014]

Sol. Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}(2x\sqrt{1-x^2})$

We have to determine $\frac{du}{dv}$.

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

Now, $$u = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow u = \tan^{-1}(\cot\theta) \Rightarrow u = \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \theta\right)\right]$$

$$\Rightarrow u = \frac{\pi}{2} - \theta \Rightarrow u = \frac{\pi}{2} - \sin^{-1}x$$

$$\Rightarrow \frac{du}{dx} = 0 - \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Again, $v = \cos^{-1}(2x\sqrt{1-x^2})$

$\because x = \sin\theta$

$$\therefore v = \cos^{-1}(2\sin\theta\sqrt{1-\sin^2\theta}) \Rightarrow v = \cos^{-1}(2\sin\theta.\cos\theta)$$

$$\Rightarrow v = \cos^{-1}(\sin 2\theta) \Rightarrow v = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$$

$$\Rightarrow v = \frac{\pi}{2} - 2\theta \Rightarrow v = \frac{\pi}{2} - 2\sin^{-1}x$$

$$\Rightarrow \frac{dv}{dx} = 0 - \frac{2}{\sqrt{1-x^2}} \Rightarrow \frac{dv}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

$$\left[\begin{array}{l} \because -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \\ \Rightarrow \sin\left(-\frac{\pi}{4}\right) < \sin\theta < \sin\left(\frac{\pi}{4}\right) \\ \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2} \\ \Rightarrow \frac{\pi}{2} > -2\theta > -\frac{\pi}{2} \\ \Rightarrow \pi > \left(\frac{\pi}{2} - 2\theta\right) > 0 \\ \Rightarrow \left(\frac{\pi}{2} - 2\theta\right) \in (0,\pi) \subset [0,\pi] \end{array}\right]$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{2}{\sqrt{1-x^2}}} = \frac{1}{2}$$

[**Note:** Here the range of x is taken as $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.]

11. Differentiate the following function with respect to x:

$$y = (\sin x)^x + \sin^{-1}\sqrt{x}$$

[*CBSE Delhi 2009, 2013, 2017*]

Sol. Given, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$y = u + v$, where $u = (\sin x)^x$, $v = \sin^{-1}\sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots(i)$$

Now, $u = (\sin x)^x$

Taking log both sides, we get

$$\log u = \log(\sin x)^x \quad \Rightarrow \quad \log u = x \,.\, \log(\sin x)$$

Differentiating both sides with respect to x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x.\frac{1}{\sin x}\cos x + \log\sin x \Rightarrow \quad \frac{du}{dx} = u\{x\cot x + \log\sin x\}$$

$$\Rightarrow \quad \frac{du}{dx} = (\sin x)^x\{x\cot x + \log\sin x\} \quad \ldots(ii)$$

Also, $v = \sin^{-1}\sqrt{x}$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}} \quad \ldots(iii)$$

From (*i*), (*ii*) and (*iii*), we get

$$\therefore \quad \frac{dy}{dx} = (\sin x)^x\{x\cot x + \log\sin x\} + \frac{1}{2\sqrt{x(1-x)}}$$

12. If $x^{13}y^7 = (x+y)^{20}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$. [*CBSE (F) 2012*]

OR

If $x^m y^n = (x+y)^{m+n}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$. [*CBSE (F) 2014*]

Sol. Given $x^{13}y^7 = (x+y)^{20}$

Taking logarithm on both sides, we get

$$\log(x^{13}y^7) = \log(x+y)^{20}$$

$$\Rightarrow \quad \log x^{13} + \log y^7 = 20\log(x+y) \quad \Rightarrow \quad 13\log x + 7\log y = 20\log(x+y)$$

Differentiating both sides with respect to x, we get

$$\frac{13}{x} + \frac{7}{y}\cdot\frac{dy}{dx} = \frac{20}{x+y}\cdot\left(1 + \frac{dy}{dx}\right) \quad \Rightarrow \quad \frac{13}{x} - \frac{20}{x+y} = \left(\frac{20}{x+y} - \frac{7}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{13x + 13y - 20x}{x(x+y)} = \left(\frac{20y - 7x - 7y}{(x+y).y}\right)\frac{dy}{dx} \quad \Rightarrow \quad \frac{13y - 7x}{x(x+y)} = \left(\frac{13y - 7x}{y(x+y)}\right)\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{13y - 7x}{x(x+y)} \times \frac{y(x+y)}{13y - 7x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

OR

Do yourself (similar question)

13. **Differentiate with respect to x:**

$$\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)$$ [CBSE (AI) 2013]

Sol. Let $y = \sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right) = \sin^{-1}\left(\frac{2.2^x.3^x}{1+(6^2)^x}\right) = \sin^{-1}\left(\frac{2.6^x}{1+(6^x)^2}\right)$

Let $6^x = \tan\theta \quad \Rightarrow \quad \theta = \tan^{-1}(6^x)$

$\therefore \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) \quad \Rightarrow \quad y = \sin^{-1}(\sin 2\theta)$

$\Rightarrow \quad y = 2\theta \quad \Rightarrow \quad y = 2.\tan^{-1}(6^x)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+(6^x)^2}.\log_e 6.6^x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2.6^x.\log_e 6}{1+36^x}$

14. **If $x\sin(a+y) + \sin a\cos(a+y) = 0$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.** [CBSE (AI) 2013]

Sol. Given $x\sin(a+y) + \sin a\cos(a+y) = 0$

$\Rightarrow \quad x = -\frac{\sin a.\cos(a+y)}{\sin(a+y)} \quad \Rightarrow \quad x = -\sin a.\cot(a+y)$

Differentiating with respect to y, we get

$$\frac{dx}{dy} = +\sin a.\text{cosec}^2(a+y) = \frac{\sin a}{\sin^2(a+y)}$$

$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

15. **If $e^x + e^y = e^{x+y}$, then prove that $\frac{dy}{dx} + e^{y-x} = 0$.** [CBSE (F) 2014]

Sol. Given, $e^x + e^y = e^{x+y}$

Differentiating both sides with respect to x, we get

$$e^x + e^y.\frac{dy}{dx} = e^{x+y}\left\{1+\frac{dy}{dx}\right\}$$

$\Rightarrow \quad e^x + e^y.\frac{dy}{dx} = e^{x+y} + e^{x+y}.\frac{dy}{dx} \quad \Rightarrow \quad (e^{x+y} - e^y)\frac{dy}{dx} = e^x - e^{x+y}$

$\Rightarrow \quad (e^x + e^y - e^y)\frac{dy}{dx} = e^x - e^x - e^y \quad [\because e^x + e^y = e^{x+y} \text{(given)}]$

$\Rightarrow \quad e^x.\frac{dy}{dx} = -e^y \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{e^y}{e^x}$

$\Rightarrow \quad \frac{dy}{dx} = -e^{y-x} \quad \Rightarrow \quad \frac{dy}{dx} + e^{y-x} = 0$

16. **If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = -\frac{y\log x}{x\log y}$.** [CBSE (East) 2016]

Sol. We have $x = e^{\cos 2t}$

$$\frac{dx}{dt} = e^{\cos 2t}(-2\sin 2t) = -2x\sin 2t \quad \text{[Differentiating w.r.t. } t\text{]}$$

Again $y = e^{\sin 2t}$

$$\frac{dy}{dt} = e^{\sin 2t}.2\cos 2t = 2y\cos 2t \quad \text{[Differentiating w.r.t. } t\text{]}$$

Now, $\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \dfrac{2y\cos 2t}{-2x\sin 2t} \Rightarrow \dfrac{dy}{dx} = \dfrac{-y\cos 2t}{x\sin 2t}$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{y\log x}{x\log y}$ $[\because x = e^{\cos 2t} \Rightarrow \log x = \cos 2t;\ y = e^{\sin 2t} \Rightarrow \log y = \sin 2t]$

Hence proved.

17. If $x \in R - [-1, 1]$ then prove that the derivative of $\sec^{-1}x$ with respect to x is $\dfrac{1}{|x|\sqrt{x^2-1}}$. ***[HOTS]***

Sol. Let $y = \sec^{-1} x$

Then, $\sec y = \sec(\sec^{-1} x) = x$

Differentiating both sides with respect to x, we have

$\Rightarrow \dfrac{d}{dx}\sec y = \dfrac{d}{dx}(x) \Rightarrow \dfrac{d}{dy}(\sec y)\dfrac{dy}{dx} = 1$

$\Rightarrow \sec y \tan y \dfrac{dy}{dx} = 1$ [Using chain rule]

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sec y \tan y} = \dfrac{1}{|\sec y| . |\tan y|}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{|\sec y|\sqrt{\tan^2 y}} = \dfrac{1}{|\sec y|\sqrt{\sec^2 y - 1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$

[If $x > 1$, then $y \in \left(0, \dfrac{\pi}{2}\right)$
$\therefore \sec y > 0, \tan y > 0$
$\Rightarrow |\sec y| . |\tan y| = \sec y \tan y$
If $x < -1$, then
$y \in \left(\dfrac{\pi}{2}, \pi\right) \therefore \sec y < 0, \tan y < 0$
$\Rightarrow |\sec y|\,|\tan y|$
$\Rightarrow (-\sec y)(-\tan y) = \sec y \tan y$]

18. If $y = e^{x^2\cos x} + (\cos x)^x$, then find $\dfrac{dy}{dx}$. ***[CBSE 2020 (65/1/1)]***

Sol. Given, $y = e^{x^2\cos x} + (\cos x)^x$

Let $u = e^{x^2\cos x}$ and $v = (\cos x)^x$

$\therefore \quad y = u + v \Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ ...(*i*)

Now, $u = e^{x^2\cos x}$

Taking log on both sides, we have

$$\log u = \log e^{x^2\cos x} = x^2\cos x$$

Differentiating w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x^2 \times (-\sin x) + \cos x \times 2x$$

$\Rightarrow \dfrac{du}{dx} = u(-x^2\sin x + 2x\cos x)$

$\therefore \dfrac{du}{dx} = e^{x^2\cos x}(-x^2\sin x + 2x\cos x)$

Again, $v = (\cos x)^x$

Taking log on both sides, we have

$$\log v = x\log\cos x$$

Differentiating w.r.t. x, we have

$$\frac{1}{v}\frac{dv}{dx} = x \times \frac{1}{\cos x} \times (-\sin x) + \log\cos x$$

$$= -x\tan x + \log\cos x$$

$\therefore \quad \frac{dv}{dx} = v(-x\tan x + \log\cos x)$

$\qquad = (\cos x)^x(-x\tan x + \log\cos x)$

Putting values in equation (*i*), we have

$$\frac{dy}{dx} = e^{x^2\cos x}(-x^2\sin x + 2x\cos x) + (\cos x)^x(-x\tan x + \log\cos x)$$

19. Differentiate $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, w.r.t. $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$. *[CBSE 2023 (65/5/1)]*

Sol. Let $y = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, $x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$y = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right) = \sec^{-1}\left(\frac{1}{\cos\theta}\right) = \sec^{-1}(\sec\theta)$$

$$y = \theta = \sin^{-1}x$$

Also, $t = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta)$

$= 2\theta = 2\sin^{-1}x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \frac{dt}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dy}{dt} = \frac{dy/dx}{dt/dx} = \frac{1/\sqrt{1-x^2}}{2/\sqrt{1-x^2}} = \frac{1}{2}$$

Second Order Derivatives

1. If $x = a\sec^3\theta$, $y = a\tan^3\theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$. *[CBSE 2020 (65/1/2)]*

Sol.

$x = a\sec^3\theta$

Differentiating with respect to x

$\Rightarrow \frac{dx}{d\theta} = 3a\sec^2\theta\sec\theta\tan\theta = 3a\sec^3\theta\tan\theta$

$y = a\tan^3\theta$

Differentiating with respect to y

$\Rightarrow \frac{dy}{d\theta} = 3a\tan^2\theta\sec^2\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta$ — (i)

Differenting equation (i) with respect to x

$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\sin\theta\frac{d\theta}{dx}$

$\frac{d^2y}{dx^2} = \frac{\cos\theta}{3a\sec^3\theta\tan\theta} = \frac{\cos\theta\cos^3\theta\cos\theta}{3a\sin\theta} = \frac{\cos^5\theta}{3a\sin\theta}$

$\left(\frac{d^2y}{dx^2}\right)_{(\theta=\pi/4)} = \frac{\sqrt{2}}{(\sqrt{2})^5 3a} = \frac{1}{12a}$ Answer

[Topper's Answer 2020]

2. If $x = a(\theta - \sin\theta), y = a(1 - \cos\theta), a > 0,$ then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{3}$. [CBSE 2020 (65/1/3)]

Sol. $\frac{dy}{d\theta} = a\sin\theta, \frac{dx}{d\theta} = a(1-\cos\theta)$ ½ + ½

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$ 1

$$\frac{d^2y}{dx^2} = -\frac{1}{2}.\text{cosec}^2\frac{\theta}{2}.\frac{d\theta}{dx} = -\frac{\text{cosec}^2\frac{\theta}{2}}{2a(1-\cos\theta)}$$ 1

$$\left.\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{3}} = -\frac{1}{2}\times\frac{4}{a\left(1-\frac{1}{2}\right)} = -\frac{4}{a}$$ 1

[CBSE Marking Scheme 2020 (65/1/3)]

3. If $y = e^{a\cos^{-1}x}, -1 < x < 1,$ then show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$. [CBSE 2020 (65/2/1)]

Sol. $\frac{dy}{dx} = \frac{-ae^{a\cos^{-1}x}}{\sqrt{1-x^2}}$ 1

$$\sqrt{1-x^2}\frac{dy}{dx} = -ae^{a\cos^{-1}x}$$ ½

Differentiating again & getting

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}.\frac{dy}{dx} = \frac{a^2e^{a\cos^{-1}x}}{\sqrt{1-x^2}}$$ 2

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$ ½

[CBSE Marking Scheme 2020 (65/2/1)]

4. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, then show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

[CBSE (F) 2014; Delhi 2015]

Sol. Given, $x = a\cos\theta + b\sin\theta \Rightarrow \frac{dx}{d\theta} = -a\sin\theta + b\cos\theta$ ...(*i*)

Also, $y = a\sin\theta - b\cos\theta \Rightarrow \frac{dy}{d\theta} = a\cos\theta + b\sin\theta$ ...(*ii*)

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\cos\theta + b\sin\theta}{-a\sin\theta + b\cos\theta}$$ [From (*i*) and (*ii*)]

$$\Rightarrow \quad \frac{dy}{dx} = \frac{a\cos\theta + b\sin\theta}{b\cos\theta - a\sin\theta} \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$ ...(*iii*)

Differentiating again with respect to *x*, we get

$$\frac{d^2y}{dx^2} = -\frac{y - x.\frac{dy}{dx}}{y^2} \Rightarrow y^2\frac{d^2y}{dx^2} = -y + x\frac{dy}{dx} \Rightarrow y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$$

5. If $y = Pe^{ax} + Qe^{bx}$, then show that $\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$. **[CBSE (AI) 2014]**

Sol.

13. $y = Pe^{ax} + Qe^{bx} \rightarrow (1)$

$\frac{dy}{dx} = aPe^{ax} + bQe^{bx} \rightarrow (2)$

From (1),

$Pe^{ax} = y - Qe^{bx}$

Substituting in (2),

$\frac{dy}{dx} = a(y - Qe^{bx}) + bQe^{bx}$

$\frac{d^2y}{dx^2} = a^2Pe^{ax} + b^2Qe^{bx} \rightarrow (3)$

Required to prove:

$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$

Evaluating LHS,

$LHS = \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby$

$= a^2Pe^{ax} + b^2Qe^{bx} - (a+b)\left[aPe^{ax} + bQe^{bx}\right] - ab\left(Pe^{ax} + Qe^{bx}\right)$ [From (1), (2) and (3)]

$= a^2Pe^{ax} + b^2Qe^{bx} - a^2Pe^{ax} - b^2Qe^{bx} - abQe^{bx} - abPe^{ax} - abPe^{ax} - abQe^{bx}$

$= 0$ (Cancelling all terms)

$RHS = 0$

Hence $LHS = RHS$

$\therefore \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$ Hence, proved.

[Topper's Answer 2014]

6. If $y = \sin(\log x)$, then prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$. **[CBSE (F) 2013]**

Sol. Given, $y = \sin(\log x)$

$$\Rightarrow \quad \frac{dy}{dx} = \cos(\log x) \times \frac{1}{x} = \frac{\cos(\log x)}{x}$$

Again,
$$\frac{d^2y}{dx^2} = \frac{x\left[-\sin(\log x) \times \frac{1}{x}\right] - \cos(\log x)}{x^2} = \frac{-\cos(\log x) - \sin(\log x)}{x^2}$$

Now, LHS $= x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y$

$$= \frac{x^2\{-\cos(\log x) - \sin(\log x)\}}{x^2} + \frac{x\cos(\log x)}{x} + \sin(\log x)$$

$$= -\cos(\log x) - \sin(\log x) + \cos(\log x) + \sin(\log x) = 0 = \text{RHS}$$

7. If $y = \text{cosec}^{-1} x, x > 1$, then show that $x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0.$ *[CBSE (AI) 2010]*

Sol. $\because \quad y = \text{cosec}^{-1} x$

Differentiating with respect to x, we get

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{x\sqrt{x^2-1}.0 + 1.\left\{x.\frac{2x}{2\sqrt{x^2-1}} + \sqrt{x^2-1}\right\}}{x^2(x^2-1)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{x^2+x^2-1}{x^2(x^2-1).\sqrt{x^2-1}} = \frac{2x^2-1}{\sqrt{x^2-1}.x^2(x^2-1)}$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2} = \frac{2x^2-1}{x\sqrt{x^2-1}} = (2x^2-1)\left(-\frac{dy}{dx}\right)$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2} + (2x^2-1)\frac{dy}{dx} = 0$$

8. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$ *[CBSE Delhi 2009, 2012]*

Sol. Given, $y = 3\cos(\log x) + 4\sin(\log x)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x} \quad \Rightarrow \quad y_1 = \frac{1}{x}[-3\sin(\log x) + 4\cos(\log x)]$$

Again differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{x\left[\frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x}\right] - [-3\sin(\log x) + 4\cos(\log x)]}{x^2}$$

$$= \frac{-3\cos(\log x) - 4\sin(\log x) + 3\sin(\log x) - 4\cos(\log x)}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\sin(\log x) - 7\cos(\log x)}{x^2} \quad \Rightarrow \quad y_2 = \frac{-\sin(\log x) - 7\cos(\log x)}{x^2}$$

Now, LHS $= x^2y_2 + xy_1 + y$

$$= x^2\left(\frac{-\sin(\log x) - 7\cos(\log x)}{x^2}\right) + x \times \frac{1}{x}[-3\sin(\log x) + 4\cos(\log x)] + 3\cos(\log x) + 4\sin(\log x)$$

$$= -\sin(\log x) - 7\cos(\log x) - 3\sin(\log x) + 4\cos(\log x) + 3\cos(\log x) + 4\sin(\log x)$$

$= 0 =$ RHS

9. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, $0 < t < \frac{\pi}{2}$, find $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

[CBSE (AI) 2012]

Sol. Given, $x = a(\cos t + t\sin t)$

Differentiating both sides with respect to t, we get

$$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t) \quad \Rightarrow \quad \frac{dx}{dt} = at\cos t \qquad ...(i)$$

Differentiating again with respect to t, we get

$$\frac{d^2x}{dt^2} = a(-t\sin t + \cos t) = a(\cos t - t\sin t)$$

Again, $y = a(\sin t - t\cos t)$

Differentiating with respect to t, we get

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) \quad \Rightarrow \quad \frac{dy}{dt} = at\sin t \qquad ...(ii)$$

Differentiating again with respect to t, we get

$$\frac{d^2y}{dt^2} = a(t\cos t + \sin t)$$

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ [From (i) and (ii)]

$$\Rightarrow \quad \frac{dy}{dx} = \frac{at\sin t}{at\cos t} \quad \Rightarrow \quad \frac{dy}{dx} = \tan t$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t.\frac{dt}{dx} = \sec^2 t.\frac{1}{dx/dt} = \frac{\sec^2 t}{at\cos t} = \frac{\sec^3 t}{at} \qquad \text{[From (i)]}$$

Hence, $\frac{d^2x}{dt^2} = a(\cos t - t\sin t)$, $\frac{d^2y}{dt^2} = a(t\cos t + \sin t)$ and $\frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}$.

10. If $y = \log\left[x + \sqrt{x^2 + a^2}\right]$, show that $(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$. *[CBSE Delhi 2013]*

Sol. Given $y = \log\left[x + \sqrt{x^2 + a^2}\right]$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}}.\left[1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right] = \frac{x + \sqrt{x^2 + a^2}}{(x + \sqrt{x^2 + a^2})(\sqrt{x^2 + a^2})}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}} \qquad ...(i)$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = -\frac{1}{2}(x^2 + a^2)^{-\frac{3}{2}}.2x = \frac{-x}{(x^2 + a^2)^{\frac{3}{2}}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-x}{(x^2 + a^2).\sqrt{x^2 + a^2}} \quad \Rightarrow \quad (x^2 + a^2)\frac{d^2y}{dx^2} = -\frac{x}{\sqrt{x^2 + a^2}}$$

$$\Rightarrow \quad (x^2 + a^2)\frac{d^2y}{dx^2} + x.\frac{dy}{dx} = 0 \qquad \text{[From (i)]}$$

11. If $y = x^x$, then prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$. *[CBSE Delhi 2014, 2016]*

Sol. Given, $y = x^x$

Taking logarithm on both sides, we get

$$\log y = x \cdot \log x$$

Differentiating both sides, we get

$$\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \qquad \Rightarrow \quad \frac{dy}{dx} = y(1 + \log x) \qquad \ldots(i)$$

Again differentiating both sides, we get

$$\frac{d^2y}{dx^2} = y \cdot \frac{1}{x} + (1 + \log x) \cdot \frac{dy}{dx} \qquad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y} \cdot \frac{dy}{dx} \cdot \frac{dy}{dx} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 \qquad \Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$

12. If $y = \left(x + \sqrt{1 + x^2}\right)^n$ then show that $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y$. *[CBSE (F) 2015]*

Sol. Given $y = \left(x + \sqrt{1 + x^2}\right)^n$

Differentiating with respect to x, we get

$$\Rightarrow \quad \frac{dy}{dx} = n\left(x + \sqrt{1 + x^2}\right)^{n-1} \cdot \left\{1 + \frac{2x}{2\sqrt{1 + x^2}}\right\} \qquad \Rightarrow \quad \frac{dy}{dx} = n\left(x + \sqrt{1 + x^2}\right)^{n-1} \cdot \left(\frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{n\left(x + \sqrt{1 + x^2}\right)^n}{\sqrt{1 + x^2}} \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{ny}{\sqrt{1 + x^2}}$$

$$\Rightarrow \quad \sqrt{1 + x^2} \cdot \frac{dy}{dx} = ny$$

Again differentiating with respect to x, we get

$$\sqrt{1 + x^2} \cdot \frac{d^2y}{dx^2} + \frac{2x}{2\sqrt{1 + x^2}} \cdot \frac{dy}{dx} = n\frac{dy}{dx} \qquad \Rightarrow \quad (1 + x^2)\frac{d^2y}{dx^2} + x \cdot \frac{dy}{dx} = n \cdot \sqrt{1 + x^2} \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad (1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n \cdot \sqrt{1 + x^2} \cdot \frac{ny}{\sqrt{1 + x^2}} \qquad \Rightarrow \quad (1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y$$

13. If $y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ then prove that $x(x + 1)^2y_2 + (x + 1)^2y_1 = 2$. *[CBSE Sample Paper 2018]*

Sol. $\because$

$$y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 \qquad \Rightarrow \quad y = 2\log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = 2\log\left(\frac{x + 1}{\sqrt{x}}\right)$$

$$y = 2\log(x + 1) - 2\log\sqrt{x} \qquad \Rightarrow \quad y = 2\log(x + 1) - \log x$$

$$\Rightarrow \quad y_1 = \frac{2}{x + 1} - \frac{1}{x} = \frac{2x - x - 1}{x(x + 1)} \qquad \Rightarrow \quad y_1 = \frac{x - 1}{x(x + 1)}$$

$$\Rightarrow \quad y_2 = \frac{x(x + 1) - (x - 1)(2x + 1)}{x^2(x + 1)^2} \qquad \Rightarrow \quad y_2 = \frac{x^2 + x - 2x^2 - x + 2x + 1}{x^2(x + 1)^2}$$

$$\Rightarrow \quad y_2 = \frac{-x^2 + 2x + 1}{x^2(x + 1)^2}$$

Now, $x(x+1)^2 y_2 + (x+1)^2 y_1 = x(x+1)^2 . \frac{-x^2+2x+1}{x^2(x+1)^2} + (x+1)^2 . \frac{(x-1)}{x(x+1)}$

$$= \frac{-x^2+2x+1}{x} + \frac{(x+1)(x-1)}{x}$$

$$= \frac{-x^2+2x+1+x^2-1}{x} = \frac{2x}{x} = 2$$

Hence proved.

14. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0$. *[CBSE 2018]*

Sol. $y = \sin(\sin x)$

$$\Rightarrow \frac{dy}{dx} = \cos(\sin x)\frac{d}{dx}(\sin x) \quad \Rightarrow \quad \frac{dy}{dx} = \cos(\underset{\text{I}}{\sin x})\underset{\text{II}}{\cos x}$$

Again differentiating w.r.t x on both sides, we get

$$\frac{d^2y}{dx^2} = \cos(\sin x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}\cos(\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \cos(\sin x)(-\sin x) - (\cos x)\{(\sin(\sin x)\}(\cos x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\sin x\cos(\sin x) - \cos^2 x\sin(\sin x)$$

Putting these values in LHS, we get

$$\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x$$

$$= \{-\sin x\cos(\sin x) - \cos^2 x\sin(\sin x)\} + \tan x\{\cos x\cos(\sin x)\} + y\cos^2 x$$

$$= -\sin x\cos(\sin x) - \cos^2 x\sin(\sin x) + \tan x\cos x\cos(\sin x) + y\cos^2 x$$

$$= -\sin x\cos(\sin x) - \cos^2 x\sin(\sin x) + \frac{\sin x}{\cos x}\cos x\cos(\sin x) + \sin(\sin x)\cos^2 x = 0$$

Hence proved.

15. If $x = \sin t$ and $y = \sin pt$ then prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$. *[NCERT Exemplar]*

Sol. We have, $x = \sin t$ and $y = \sin pt$

$$\therefore \quad \frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = \cos pt.p \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p.\cos pt}{\cos t} \quad ...(i)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\cos t.\frac{d}{dt}(p.\cos pt)\frac{dt}{dx} - p\cos pt.\frac{d}{dt}\cos t.\frac{dt}{dx}}{\cos^2 t}$$

$$= \frac{[\cos t.p.(-\sin pt).p - p\cos pt.(-\sin t)]\frac{dt}{dx}}{\cos^2 t} = \frac{[-p^2\sin pt.\cos t + p\sin t.\cos pt]\frac{1}{\cos t}}{\cos^2 t}$$

$$\frac{d^2y}{dx^2} = \frac{-p^2\sin pt.\cos t + p\cos pt.\sin t}{\cos^3 t} \quad ...(ii)$$

Since, we have to prove

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$$

$$\therefore \quad \text{LHS} = (1-\sin^2 t)\frac{[-p^2\sin pt.\cos t + p\cos pt.\sin t]}{\cos^3 t} - \sin t.\frac{p\cos pt}{\cos t} + p^2\sin pt$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}(1-\sin^2 t)(-p^2 \sin pt.\cos t + p\cos pt.\sin t) \\ -p\cos pt.\sin t.\cos^2 t + p^2 \sin pt.\cos^3 t\end{bmatrix}$$

$$= \frac{1}{\cos^3 t}\begin{bmatrix}-p^2 \sin pt.\cos^3 t + p\cos pt.\sin t.\cos^2 t \\ -p\cos pt.\sin t.\cos^2 t + p^2 \sin pt.\cos^3 t\end{bmatrix} \qquad [\because 1-\sin^2 t = \cos^2 t]$$

$$= \frac{1}{\cos^3 t}.0 = 0$$

Hence proved.

16. If $y = \tan x + \sec x$, then prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}$. [*CBSE 2023 (65/5/1)*]

Sol. $y = \tan x + \sec x = \frac{\sin x}{\cos x} + \frac{1}{\cos x} = \frac{\sin x + 1}{\cos x}$

$$\frac{dy}{dx} = \frac{\cos x(\cos x + 0) - (1+\sin x)\times(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin x(1+\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin x + \cos^2 x}{\cos^2 x}$$

$$= \frac{1+\sin x}{1-\sin^2 x} \qquad \begin{bmatrix}\because \sin^2 x + \cos^2 x = 1 \\ \text{and } \cos^2 x = 1-\sin^2 x\end{bmatrix}$$

$$= \frac{(1+\sin x)}{(1-\sin x)(1+\sin x)} = \frac{1}{1-\sin x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1-\sin x}\right) = -\frac{1}{(1-\sin x)^2}\times(0-\cos x)$$

$$= \frac{\cos x}{(1-\sin x)^2}$$

17. If $y = (\tan^{-1} x)^2$, show that $(x^2+1)^2 y_2 + 2x(x^2+1)y_1 = 2$. [*CBSE (AI) 2012; Delhi 2012*]

Sol. We have, $y = (\tan^{-1} x)^2$ ...(*i*)

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2\tan^{-1} x.\frac{1}{1+x^2} \qquad ...(ii)$$

or $(1+x^2)\, y_1 = 2\tan^{-1} x$ $\left[\text{where } y_1 = \frac{dy}{dx}\right]$

Again differentiating with respect to x, we get

$$(1+x^2).\frac{dy_1}{dx} + y_1\frac{d}{dx}(1+x^2) = 2.\frac{1}{1+x^2}$$

$$\Rightarrow \quad (1+x^2).y_2 + y_1.2x = \frac{2}{1+x^2}$$

or $(1+x^2)^2 y_2 + 2x\,(1+x^2)\, y_1 = 2$ $\left[\text{where } y_2 = \frac{d^2y}{dx^2} \text{ and } y_1 = \frac{dy}{dx}\right]$

18. If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, then prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$. Hence show that $\sin a\frac{d^2y}{dx^2} + \sin 2(a+y)\frac{dy}{dx} = 0$. [*CBSE (F) 2014; (North) 2016*]

Sol. Given, $\cos y = x\cos(a+y)$

$$\therefore \quad x = \frac{\cos y}{\cos(a+y)}$$

Differentiating with respect to y on both sides, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\times(-\sin y) - \cos y \times [-\sin(a+y)]}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos y \sin(a+y) - \sin y \cos(a+y)}{\cos^2(a+y)} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\cos^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{\sin a}\left\{-2\cos(a+y).\sin(a+y).\frac{dy}{dx}\right\}$$

$$\Rightarrow \quad \sin a\frac{d^2y}{dx^2} = -\sin 2(a+y).\frac{dy}{dx} \quad \Rightarrow \quad \sin a\frac{d^2y}{dx^2} + \sin 2(a+y).\frac{dy}{dx} = 0$$

Questions for Practice

■ Objective Type Questions

1. Choose and write the correct option in each of the following questions.

(*i*) If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$, then which of the following can be a discontinuous function?

(*a*) $f(x) + g(x)$ (*b*) $f(x) - g(x)$ (*c*) $f(x) . g(x)$ (*d*) $\frac{g(x)}{f(x)}$

(*ii*) The set of points where the function f given by $f(x) = |2x - 1|\sin x$ is differentiable equals [*NCERT Exemplar*]

(*a*) R (*b*) $R - \left\{\frac{1}{2}\right\}$ (*c*) $(0, \infty)$ (*d*) none of these

(*iii*) Let $f(x) = |\cos x|$. Then,

(*a*) f is everywhere differentiable.

(*b*) f is everywhere continuous but not differentiable at $x = n\pi, n \in Z$.

(*c*) f is everywhere continuous but not differentiable at $x = (2n+1)\frac{\pi}{2}, n \in Z$.

(*d*) none of these.

(*iv*) If $y = A e^{5x} + B e^{-5x}$, then $\frac{d^2y}{dx^2}$ is equal to [*CBSE 2020 (65/5/1)*]

(*a*) $25y$ (*b*) $5y$ (*c*) $-25y$ (*d*) $15y$

(*v*) The derivative of x^2 w.r.t x is

(*a*) x^{2x-1} (*b*) $2x^{2x}\log x$ (*c*) $2x^{2x}(1+\log x)$ (*d*) $2x^2(1-\log x)$

(*vi*) If $f'(1) = 2$ and $y = f(\log_e x)$, then $\frac{dy}{dx}$ at $x = e$ is

(*a*) 0 (*b*) 1 (*c*) e (*d*) $\frac{2}{e}$

Conceptual Questions

2. If $f(x)=\begin{cases}\dfrac{\sin^{-1}x}{x}, & x\neq 0\\ k, & x=0\end{cases}$, is continuous at $x = 0$, then write the value of k.

3. Determine the value of 'k' for which the following function is continuous at $x = 3$:

$$f(x)=\begin{cases}\dfrac{(x+3)^2-36}{x-3}, & x\neq 3\\ k, & x=3\end{cases}$$ **[CBSE (AI) 2017]**

4. If $y=\left[\sin\frac{x}{2}+\cos\frac{x}{2}\right]^2$, find $\frac{dy}{dx}$ at $x=\frac{\pi}{6}$.

5. Find $\frac{dy}{dx}$, if $y=\tan^{-1}\left[\frac{1+\tan x}{1-\tan x}\right]$, $x\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$.

6. If $y = \log(e^x)$, then find $\frac{dy}{dx}$. **[CBSE 2019 (65/4/3)]**

Very Short Answer Questions

7. Find k if $f(x)$ is continuous at $x = 0$.

$$f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, & \text{if } x\neq 0\\ k, & \text{if } x=0\end{cases}$$

8. Examine the continuity at the indicated point.

$f(x) = |x| + |x-1|$ at $x = 1$

9. If $f(x) = |\cos x - \sin x|$, find $f'(\pi/6)$.

10. If $y = 5\cos x - 3\sin x$, prove that $\frac{d^2y}{dx^2}+y=0$.

11. Find $\frac{dy}{dx}$ at $t=\frac{2\pi}{3}$ when $x = 10(t - \sin t)$ and $y = 12(1 - \cos t)$. **[CBSE (F) 2017]**

12. Find $\frac{dy}{dx}$ at $x=1$, $y=\frac{\pi}{4}$ if $\sin^2 y+\cos xy=K$. **[CBSE Delhi 2017]**

Short Answer Questions

13. Show that the function 'f' defined by $f(x)=\begin{cases}3x-2, & 0<x\leq 1\\ 2x^2-x, & 1<x\leq 2\\ 5x-4, & x>2\end{cases}$ is continuous at $x = 2$, but not differentiable. **[CBSE Delhi 2010]**

14. For what value of k is the following function continuous at $x=-\frac{\pi}{6}$? **[CBSE Sample Paper 2017]**

$$f(x)=\begin{cases}\dfrac{\sqrt{3}\sin x+\cos x}{x+\frac{\pi}{6}}, & x\neq -\frac{\pi}{6}\\ k, & x=-\frac{\pi}{6}\end{cases}$$

15. Discuss the differentiability of the function $f(x)=\begin{cases}2x-1, & x<\frac{1}{2}\\ 3-6x, & x\geq\frac{1}{2}\end{cases}$ at $x=\frac{1}{2}$. **[CBSE Sample Paper 2017]**

16. Show that the function f given by

$$f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1}, & \text{if } x\neq 0\\ -1 & \text{if } x=0\end{cases}$$

is discontinuous at $x = 0$. **[CBSE (East) 2016]**

17. Find the values of a and b, if the function f is defined by **[CBSE (F) 2016]**

$$f(x) = \begin{cases} x^2 + 3x + a, & x \le 1 \\ bx + 2, & x > 1 \end{cases}$$

is differentiable at $x = 1$.

18. Let $f(x) = x - |x - x^2|$, $x \in [-1, 1]$. Find the point of discontinuity, (if any), of this function on $[-1, 1]$. **[CBSE Bhubaneshwar 2015]**

19. Show that the function $f(x) = |x - 1| + |x + 1|$, for all $x \in R$, but is not differentiable at the points $x = -1$ and $x = 1$. **[CBSE Allahabad 2015]**

20. If $y = \sqrt{x^2 + 1} - \log\left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}}\right)$, then find $\frac{dy}{dx}$. **[CBSE Delhi 2008]**

21. If $x\sqrt{1+y} + y\sqrt{1+x} = 0, -1 < x < 1, x \ne y$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$. **[CBSE (F) 2012]**

22. Differentiate the following function with respect to $x : (x)^{\cos x} + (\sin x)^{\tan x}$ **[CBSE Delhi 2009]**

23. If $xy = e^{(x-y)}$, then show that $\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$. **[CBSE (F) 2017]**

24. If $\frac{x}{x-y} = \log \frac{a}{x-y}$, then prove that $\frac{dy}{dx} = 2 - \frac{x}{y}$. **[CBSE Guwahati 2015]**

25. Let $y = (\log x)^x + x^{x\cos x}$, then find $\frac{dy}{dx}$. **[CBSE Sample Paper 2016]**

26. If $f(x) = \sqrt{x^2 + 1}$; $g(x) = \frac{x+1}{x^2+1}$ and $h(x) = 2x - 3$, then find $f'[h'\{g'(x)\}]$. **[CBSE Allahabad 2015]**

27. If $y = x^3 (\cos x)^x + \sin^{-1}\sqrt{x}$, find $\frac{dy}{dx}$. **[CBSE 2020 (65/3/1)]**

28. Differentiate $(\sin 2x)^x + \sin^{-1}\sqrt{3x}$ with respect to x. **[CBSE (South) 2016]**

29. Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left[\frac{6x - 4\sqrt{1-4x^2}}{5}\right]$. **[CBSE (North) 2016]**

30. If $y = \sin^{-1}(6x\sqrt{1-9x^2}), -\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$ then find $\frac{dy}{dx}$. **[CBSE Delhi 2017]**

31. If $y = \cos^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$, then find $\frac{dy}{dx}$. **[CBSE (F) 2010]**

32. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$ with respect to $\cos^{-1} x^2$. **[CBSE (South) 2016, 2019(65/4/1)]**

33. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2} - 1}{x}\right]$ with respect to x. **[CBSE (AI) 2012]**

34. Differentiate $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$. **[CBSE Delhi 2014]**

35. If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right)$, $x^2 \le 1$, then find $\frac{dy}{dx}$. **[CBSE Delhi 2015]**

36. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2} - 1}{x}\right)$ w.r.t. $\sin^{-1}\frac{2x}{1+x^2}$, if $x \in (-1, 1)$. **[CBSE (F) 2016]**

37. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$. [CBSE Delhi 2017]

38. If $y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, show that $\frac{dy}{dx} = \sec x$. Also find the value of $\frac{d^2y}{dx^2}$ at $x = \frac{\pi}{4}$. [CBSE (F) 2010]

39. If $x = \tan\left(\frac{1}{a}\log y\right)$, show that: [CBSE (AI) 2011]

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0$$

40. If $y = e^x(\sin x + \cos x)$, then show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$. [CBSE (AI) 2009]

41. If $y = e^{a\cos^{-1}x}, -1 \le x \le 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$. [CBSE (F) 2012]

42. If $(ax+b)e^{y/x} = x$, then show that $x^3\left(\frac{d^2y}{dx^2}\right) = \left(x\frac{dy}{dx} - y\right)^2$. [CBSE Ajmer 2015]

43. If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$. [CBSE (AI) 2017]

44. If $x = a(\theta - \sin\theta)$, $y = a(1+\cos\theta)$, then find $\frac{d^2y}{dx^2}$. [CBSE Delhi 2011]

45. If $y = 2\cos(\log x) + 3\sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$. [CBSE (Central) 2016]

46. If $x = a(\cos 2t + 2t\sin 2t)$ and $y = a(\sin 2t - 2t\cos 2t)$, then find $\frac{d^2y}{dx^2}$. [CBSE (Ajmer) 2015]

Answers

1. (*i*) (*d*) (*ii*) (*b*) (*iii*) (*c*) (*iv*) (*a*) (*v*) (*c*) (*vi*) (*d*)

2. $k = 1$ 3. $k = 12$ 4. $\frac{\sqrt{3}}{2}$ 5. 1 6. 1 7. $k = 2$

8. Discontinuous 9. $-\frac{1}{2}(1+\sqrt{3})$ 11. $\frac{2\sqrt{3}}{5}$ 12. $\frac{\pi}{4(\sqrt{2-1})}$

14. $k = 2$ 15. Not differentiable 17. $a = 3, b = 5$

18. No point of discontinuity. [**Hint:** $f(x) = \begin{cases} 2x - x^2, & -1 \le x < 0 \\ 0, & x = 0 \\ x^2, & 0 < x \le 1 \end{cases}$

20. $\frac{\sqrt{x^2+1}}{x}$ 22. $x^{\cos x}\left(\frac{\cos x}{x} - \sin x \log x\right) + (\sin x)^{\tan x}[1 + \sec^2 x \log \sin x]$

24. **Hint:** $\frac{x}{x-y} = \log a - \log(x-y)$ then differentiate.

25. $(\log x)^x\left\{\frac{1}{\log x} + \log(\log x)\right\} + x^{x\cos x}\{\cos x + \cos x(\log x) - x\sin x \log x\}$

26. $\frac{2\sqrt{5}}{5}$; **Hint:** At first find $f'(x)$, $g'(x)$ and $h'(x)$ and then find $f'[h'\{g'(x)\}] = f'\left[h'\left\{\frac{-x^2-2x+1}{x^2+1}\right\}\right]$

27. $x^3(\cos x)^x\left[\frac{3}{x} - x\tan x + \log(\cos x)\right] + \frac{1}{2\sqrt{x-x^2}}$

28. $(\sin 2x)^x\{2x\cot 2x + \log(\sin 2x)\} + \frac{3}{2\sqrt{3x-9x^2}}$

29. $\frac{2}{\sqrt{1-4x^2}}$

30. $\frac{6}{\sqrt{1-9x^2}}$

31. $\frac{-2^{x+1}.\log_e 2}{1+4^x}$

32. $-\frac{1}{2}$

33. $\frac{1}{2(1+x^2)}$

34. $\frac{1}{2}$

35. $\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^4}}$; [**Hint:** At first simplify $y = \tan^{-1}\left(\frac{1+\sqrt{1-x^2}}{x^2}\right)$ by multiplying with $\sqrt{1+x^2}+\sqrt{1-x^2}$.

Then let $x^2 = \sin\theta$ and then solve.]

36. $\frac{1}{4}$

38. $\sqrt{2}$

42. **Hint:** $e^{y/x} = \frac{x}{ax+b} \Rightarrow \frac{y}{x} = \log\left(\frac{x}{ax+b}\right) \Rightarrow y = \log\left(\frac{x}{ax+b}\right)^x$

44. $-\cot\frac{\theta}{2}$

46. $\frac{\sec^3 2t}{2at}$